

Applications of (In)Equational Systems: Nominal Equational Logic ~~and Nominal Rewriting~~

Chung-Kil Hur
Joint work with Marcelo Fiore

Computer Laboratory
University of Cambridge

CANS Meeting
25th October 2007

Outline of the Talk

- ① **Definition** and **Theory** of (In)Equational System.
- ② Application of Equational System: **Nominal Equational Logic**
- ③ Application of Inequational System: **Nominal Rewriting**

Overview: Definition of (In)Equational System

- 1 **Equational Systems** are a **framework** for systems with **equations**.
 - Equational Logics,
 - Computational Calculi such as λ -calculus, π -calculus,
 - Data Types, ...
- 2 **Inequational Systems** are a **framework** for **rewriting systems**.
 - First-order Term Rewriting Systems (TRS),
 - Higher-order Term Rewriting Systems (HRS),
 - Combinatory Reduction Systems (CRS),
 - Nominal Rewriting Systems (NRS), ...

Overview: Theory of (In)Equational System

For (In)Equational Systems \mathbb{S} on \mathcal{D}

- Construction of free models
- $\text{Model}(\mathbb{S})$ is monadic over \mathcal{D} .
- $\text{Model}(\mathbb{S})$ is cocomplete.

$$\begin{array}{ccc} & & \text{Model}(\mathbb{S}) \\ & \uparrow & \\ F & | & \dashv \\ & \downarrow & U \\ & & \mathcal{D} \end{array}$$

Motivation for definition: I. Signatures

Signatures as Endofunctors

- **Algebraic Theory / Term Rewriting System**

$$\Sigma_{\text{Num}} = \{ \text{zero} : 0, \text{succ} : 1, \text{plus} : 2 \}$$

- Σ_{Num} -algebra
- $D \in \mathbf{Set} / \mathbf{Pre}$
 - $\llbracket \text{zero} \rrbracket : D^0 \rightarrow D$
 - $\llbracket \text{succ} \rrbracket : D^1 \rightarrow D$
 - $\llbracket \text{plus} \rrbracket : D^2 \rightarrow D$

- **(In)Equational System**

$$\Sigma_{\text{Num}}(X) = X^0 + X^1 + X^2 \text{ on } \mathbf{Set} / \mathbf{Pre}$$

- Σ_{Num} -algebra
- $D \in \mathbf{Set} / \mathbf{Pre}$
 - $s : \Sigma_{\text{Num}} D \rightarrow D$
 $: D^0 + D^1 + D^2 \rightarrow D$

Motivation for definition: II. Equations

Equations as parallel pairs of functors

$$\{x, y\} \vdash \text{plus}(\text{succ}(x), y) = \text{succ}(\text{plus}(x, y))$$

Algebraic Theory

$$\begin{array}{ccc} 1 & D & D^2 \\ \llbracket \text{zero} \rrbracket \downarrow & \llbracket \text{succ} \rrbracket \downarrow & \llbracket \text{plus} \rrbracket \downarrow \\ D & D & D \end{array} \mapsto \forall \rho : \{x, y\} \rightarrow D$$

$\llbracket \text{plus}(\text{succ}(x), y) \rrbracket_\rho \quad \llbracket \text{succ}(\text{plus}(x, y)) \rrbracket_\rho \in D$

Equational System

$$\boxed{\Sigma_{\text{Num-Alg}}} \quad \Rightarrow \quad \boxed{(-)^2\text{-Alg}}$$
$$\begin{array}{ccc} 1 + D + D^2 & \mapsto & \begin{array}{ccc} & D^2 & \\ \llbracket \text{succ} \rrbracket \times \text{id} \swarrow & & \searrow \llbracket \text{plus} \rrbracket \\ D^2 & & D \\ \llbracket \text{plus} \rrbracket \swarrow & & \searrow \llbracket \text{succ} \rrbracket \\ & D & \end{array} \\ \llbracket \llbracket \text{zero} \rrbracket, \llbracket \text{succ} \rrbracket, \llbracket \text{plus} \rrbracket \rrbracket \downarrow & & \\ D & & \end{array}$$

Motivation for definition: II. Equations

Equations as parallel pairs of functors

$$\{x, y\} \vdash \text{plus}(\text{succ}(x), y) = \text{succ}(\text{plus}(x, y))$$

Algebraic Theory

$$\begin{array}{ccc} 1 & D & D^2 \\ \llbracket \text{zero} \rrbracket \downarrow & \llbracket \text{succ} \rrbracket \downarrow & \llbracket \text{plus} \rrbracket \downarrow \\ D & D & D \end{array} \mapsto \forall \rho : \{x, y\} \rightarrow D$$

$\llbracket \text{plus}(\text{succ}(x), y) \rrbracket_\rho \quad \llbracket \text{succ}(\text{plus}(x, y)) \rrbracket_\rho \in D$

Equational System

$$\boxed{\Sigma_{\text{Num-Alg}}} \quad \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \quad \boxed{(-)^2\text{-Alg}}$$
$$\begin{array}{ccc} 1 + D + D^2 & \xrightarrow{\quad} & \begin{array}{ccc} & D^2 & \\ \llbracket \text{succ} \rrbracket \times \text{id} \swarrow & & \searrow \llbracket \text{plus} \rrbracket \\ D^2 & & D \\ \llbracket \text{plus} \rrbracket \swarrow & & \searrow \llbracket \text{succ} \rrbracket \\ & D & \end{array} \\ \llbracket \llbracket \text{zero} \rrbracket, \llbracket \text{succ} \rrbracket, \llbracket \text{plus} \rrbracket \rrbracket \downarrow & \xrightarrow{\quad} & \end{array}$$

Motivation for definition: II. Equations

Equations as parallel pairs of functors

$$\{x, y\} \vdash \text{plus}(\text{succ}(x), y) = \text{succ}(\text{plus}(x, y))$$

Algebraic Theory

$$\begin{array}{ccc} 1 & D & D^2 \\ \llbracket \text{zero} \rrbracket \downarrow & \llbracket \text{succ} \rrbracket \downarrow & \llbracket \text{plus} \rrbracket \downarrow \\ D & D & D \end{array} \mapsto \forall \rho : \{x, y\} \rightarrow D$$
$$\llbracket \text{plus}(\text{succ}(x), y) \rrbracket_\rho \quad \llbracket \text{succ}(\text{plus}(x, y)) \rrbracket_\rho \in D$$

Equational System

$$\boxed{\Sigma_{\text{Num-Alg}}} \quad \begin{array}{c} \xrightarrow{\text{blue}} \\ \xrightarrow{\text{red}} \end{array} \quad \boxed{(-)^2\text{-Alg}}$$
$$\begin{array}{ccc} 1 + D + D^2 & \xrightarrow{\text{blue}} & \begin{array}{c} D^2 \\ \llbracket \text{succ} \rrbracket \times \text{id} \swarrow \quad \searrow \llbracket \text{plus} \rrbracket \\ D^2 \quad D \\ \llbracket \text{plus} \rrbracket \swarrow \quad \searrow \llbracket \text{succ} \rrbracket \\ D \end{array} \\ \llbracket \llbracket \text{zero} \rrbracket, \llbracket \text{succ} \rrbracket, \llbracket \text{plus} \rrbracket \rrbracket \downarrow & \xrightarrow{\text{red}} & \\ D & & \end{array}$$

Motivation for definition: II. Equations

Equations as parallel pairs of functors

$$\{x, y\} \vdash \text{plus}(\text{succ}(x), y) = \text{succ}(\text{plus}(x, y))$$

Algebraic Theory

$$\begin{array}{ccc} 1 & D & D^2 \\ \llbracket \text{zero} \rrbracket \downarrow & \llbracket \text{succ} \rrbracket \downarrow & \llbracket \text{plus} \rrbracket \downarrow \\ D & D & D \end{array} \mapsto \forall \rho : \{x, y\} \rightarrow D \quad \llbracket \text{plus}(\text{succ}(x), y) \rrbracket_{\rho} = \llbracket \text{succ}(\text{plus}(x, y)) \rrbracket_{\rho} \in D$$

Equational System

$$\boxed{\Sigma_{\text{Num-Alg}}} \xRightarrow{\quad} \boxed{(-)^2\text{-Alg}}$$
$$\begin{array}{ccc} 1 + D + D^2 & \xrightarrow{\quad} & \begin{array}{ccc} & D^2 & \\ \llbracket \text{succ} \rrbracket \times \text{id} \swarrow & & \searrow \llbracket \text{plus} \rrbracket \\ D^2 & = & D \\ \llbracket \text{plus} \rrbracket \swarrow & & \searrow \llbracket \text{succ} \rrbracket \\ & D & \end{array} \\ \llbracket \llbracket \text{zero} \rrbracket, \llbracket \text{succ} \rrbracket, \llbracket \text{plus} \rrbracket \rrbracket \downarrow & \xrightarrow{\quad} & \\ D & & \end{array}$$

Motivation for definition: II. Inequations

Rewritings as parallel pairs of functors

$$\{x, y\} \vdash \text{plus}(\text{succ}(x), y) \rightarrow \text{succ}(\text{plus}(x, y))$$

Term Rewriting System

$$\begin{array}{ccc} 1 & D & D^2 \\ \llbracket \text{zero} \rrbracket \downarrow & \llbracket \text{succ} \rrbracket \downarrow & \llbracket \text{plus} \rrbracket \downarrow \\ D & D & D \end{array} \mapsto \forall \rho : \{x, y\} \rightarrow D \quad \llbracket \text{plus}(\text{succ}(x), y) \rrbracket_{\rho} \leq \llbracket \text{succ}(\text{plus}(x, y)) \rrbracket_{\rho} \in D$$

Inequational System

$$\boxed{\Sigma_{\text{Num-Alg}}} \xrightarrow{\quad} \boxed{(-)^2\text{-Alg}}$$
$$\begin{array}{ccc} 1 + D + D^2 & \mapsto & \begin{array}{ccc} & D^2 & \\ \llbracket \text{succ} \rrbracket \times \text{id} \swarrow & & \searrow \llbracket \text{plus} \rrbracket \\ D^2 & \leq & D \\ \llbracket \text{plus} \rrbracket \swarrow & & \searrow \llbracket \text{succ} \rrbracket \\ & D & \end{array} \\ \llbracket \llbracket \text{zero} \rrbracket, \llbracket \text{succ} \rrbracket, \llbracket \text{plus} \rrbracket \rrbracket \downarrow & & \\ D & & \end{array}$$

Definition of Equational System

$$\begin{array}{ccc} \Sigma\text{-Alg} & \begin{array}{c} \xrightarrow{L} \\ \xrightarrow{R} \end{array} & \Gamma\text{-Alg} \\ U_\Sigma \downarrow & & \swarrow U_\Gamma \\ \mathcal{D} & & \end{array}$$

- **Equational System \mathbb{T}**

$$(\mathcal{D} : \Sigma \triangleright \Gamma \vdash L = R)$$

- **\mathbb{T} -Algebra**

$$(D, s : \Sigma D \rightarrow D)$$

such that

$$L(D, s) = R(D, s) : \Gamma D \rightarrow D$$

Definition of Equational System

$$\begin{array}{ccccc} \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} & \begin{array}{c} \xrightarrow{L} \\ \xrightarrow{R} \end{array} & \Gamma\text{-Alg} \\ & \searrow U_{\mathbb{T}} & \downarrow U_{\Sigma} & \swarrow U_{\Gamma} & \\ & & \mathcal{D} & & \end{array}$$

- **Equational System \mathbb{T}**

$$(\mathcal{D} : \Sigma \triangleright \Gamma \vdash L = R)$$

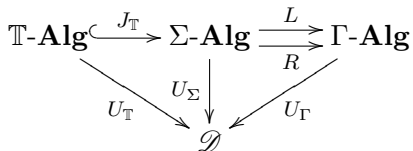
- **\mathbb{T} -Algebra**

$$(D, s : \Sigma D \rightarrow D)$$

such that

$$L(D, s) = R(D, s) : \Gamma D \rightarrow D$$

Definition of Inequational System



- Inequational System \mathbb{T} with \mathcal{D} Pre-enriched

$$(\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \leq R)$$

- \mathbb{T} -Algebra

$$(D, s : \Sigma D \rightarrow D)$$

such that

$$L(D, s) \leq R(D, s) : \Gamma D \rightarrow D$$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \leq R)$ an (In)Equational System,

$$\begin{array}{ccc} \mathbb{T}\text{-Alg} & \xleftarrow{K_{\mathbb{T}}} & \Sigma\text{-Alg} \\ & \xrightarrow{J_{\mathbb{T}}} & \\ & \searrow U_{\mathbb{T}} & \\ & & \mathcal{D} \end{array} \quad \begin{array}{c} \uparrow \\ F_{\Sigma} \end{array} \quad \begin{array}{c} \downarrow \\ U_{\Sigma} \end{array}$$

\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \leq R)$ an (In)Equational System,

$$\begin{array}{ccc} & \xleftarrow{K_{\mathbb{T}}} & \\ & \perp & \\ \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} \\ & \searrow U_{\mathbb{T}} & \uparrow F_{\Sigma} \quad \downarrow U_{\Sigma} \\ & & \mathcal{D} \end{array}$$

\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

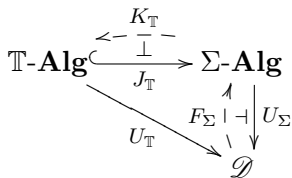
- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \leq R)$ an (In)Equational System,



\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \leq R)$ an (In)Equational System,

$$\begin{array}{ccc}
 & \xleftarrow{K_{\mathbb{T}}} & \\
 \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} \\
 & \searrow U_{\mathbb{T}} & \uparrow F_{\Sigma} \quad \downarrow U_{\Sigma} \\
 & & \mathcal{D}
 \end{array}$$

\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \leq R)$ an (In)Equational System,

$$\begin{array}{ccc}
 & \xleftarrow{K_{\mathbb{T}}} & \\
 \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} \\
 & \searrow U_{\mathbb{T}} & \uparrow F_{\Sigma} \quad \downarrow U_{\Sigma} \\
 & & \mathcal{D}
 \end{array}$$

\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \cong R)$ an (In)Equational System,

$$\begin{array}{ccc}
 & \xleftarrow{K_{\mathbb{T}}} & \\
 \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} \\
 & \searrow U_{\mathbb{T}} & \uparrow F_{\Sigma} \quad \downarrow U_{\Sigma} \\
 & & \mathcal{D}
 \end{array}$$

\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

$$\begin{array}{c}
 \Sigma X \\
 \downarrow s \\
 X \\
 \uparrow L(X,s) \quad \uparrow R(X,s) \\
 \Gamma X
 \end{array}$$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \cong R)$ an (In)Equational System,

$$\begin{array}{ccc}
 & \xleftarrow{K_{\mathbb{T}}} & \\
 \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} \\
 & \searrow U_{\mathbb{T}} & \uparrow F_{\Sigma} \quad \downarrow U_{\Sigma} \\
 & & \mathcal{D}
 \end{array}$$

\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

$$\begin{array}{ccc}
 \Sigma X & & \\
 \downarrow s & \searrow s_1 & \\
 X & \xrightarrow[e_1]{\text{coeq}} & X_1 \\
 \uparrow L(X,s) & \uparrow R(X,s) & \\
 \Gamma X & &
 \end{array}$$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \cong R)$ an (In)Equational System,

$$\begin{array}{ccc}
 & \xleftarrow{K_{\mathbb{T}}} & \\
 \mathbb{T}\text{-Alg} & \xrightarrow{J_{\mathbb{T}}} & \Sigma\text{-Alg} \\
 & \searrow U_{\mathbb{T}} & \uparrow F_{\Sigma} \quad \downarrow U_{\Sigma} \\
 & & \mathcal{D}
 \end{array}$$

\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

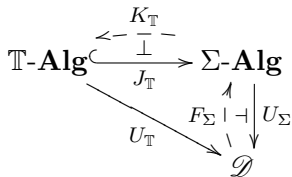
$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

$$\begin{array}{ccc}
 \Sigma X & \xrightarrow{\Sigma e_1} & \Sigma X_1 \\
 \downarrow s & \searrow s_1 & \\
 X & \xrightarrow[e_1]{\text{coeq}} & X_1 \\
 \uparrow L(X,s) & \uparrow R(X,s) & \\
 \Gamma X & &
 \end{array}$$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \cong R)$ an (In)Equational System,



\mathcal{D} is cocomplete.

Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

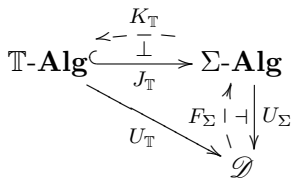
$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$

$$\begin{array}{ccccc}
 \Sigma X & \xrightarrow{\Sigma e_1} & \Sigma X_1 & & \\
 \downarrow s & \searrow s_1 & \text{po} & \searrow s_2 & \\
 X & \xrightarrow[e_1]{\text{coeq}} & X_1 & \xrightarrow{e_2} & X_2 \\
 \uparrow L(X,s) & \uparrow R(X,s) & & & \\
 \Gamma X & & & &
 \end{array}$$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \cong R)$ an (In)Equational System,



\mathcal{D} is cocomplete.

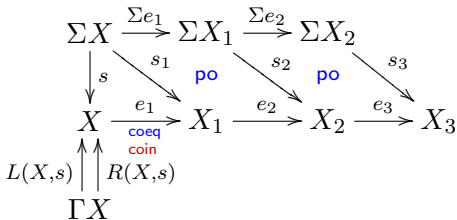
Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

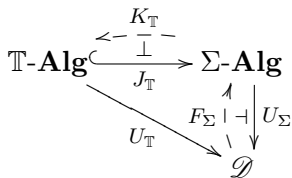
$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \cdots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$



Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \cong R)$ an (In)Equational System,



\mathcal{D} is cocomplete.

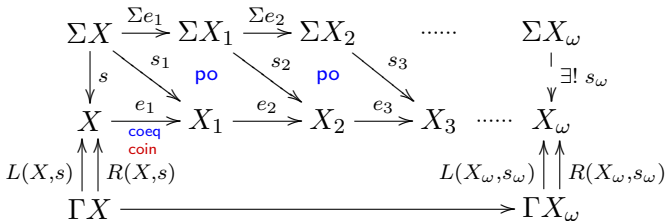
Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_\Sigma(V)$

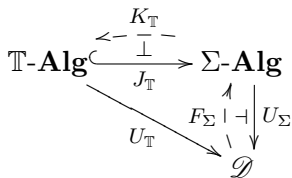
$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \dots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$



Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \cong R)$ an (In)Equational System,



\mathcal{D} is cocomplete.

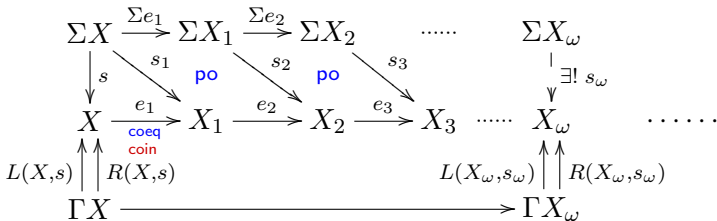
Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

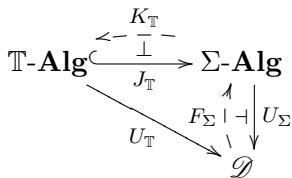
$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \dots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$



Theorem: Basic Free Construction

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L \cong R)$ an (In)Equational System,



\mathcal{D} is cocomplete.

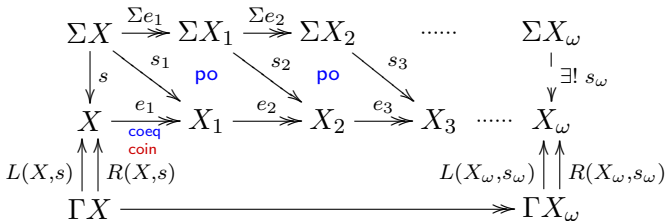
Σ, Γ preserve ω -colimits.

(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \rightarrow V + \Sigma 0 \rightarrow V + \Sigma(V + \Sigma 0) \rightarrow \dots \rightarrow (V + \Sigma(-))^* 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \rightarrow X)$



Theorem: Properties of Categories of Algebras

For $\mathbb{T} = (\mathcal{D} : \Sigma \triangleright \Gamma \vdash L = R)$ an Equational System,

$$\begin{array}{ccc} & \mathbb{T}\text{-Alg} & \\ F_{\mathbb{T}} \uparrow & \dashv & \downarrow U_{\mathbb{T}} \\ & \mathcal{D} & \\ & \downarrow M_{\mathbb{T}} = U_{\mathbb{T}}F_{\mathbb{T}} & \end{array}$$

\mathcal{D} is cocomplete.
 Σ, Γ preserve ω -colimits.

- 1 $\mathbb{T}\text{-Alg}$ is monadic over \mathcal{D} .
- 2 $\mathbb{T}\text{-Alg}$ is cocomplete.

Overview: Nominal Equational Logic

Nominal Equational Logic (NEL) is an extension of first-order equational logic with names and assertions on their freshness.

Application of Equational Systems to NEL-theories provides

- 1 **Cocompleteness** and **Monadicity** of categories of algebras.
- 2 Explicit **Construction of Free algebras**.
 - **Completeness** of Nominal Equational Logic
 - **Simplification** of NEL-proofs

Nominal Equational Logic (Clouston & Pitts '07)

A NEL-theory $\mathbb{T} = (\Sigma, E)$

$\Sigma = \{ \Sigma(n) \in \mathbf{Nom} \}_{n \in \mathbb{N}}$

$t ::= \sigma x$
| $\circ t_1 \dots t_n$

where $\sigma \in \text{Perm}(\mathbb{A})$, $x \in \mathbf{V}$,
 $\circ \in \Sigma(n)$.

E is a set of axioms of the form

$A \triangleright A_1 \# x_1, \dots, A_n \# x_n$
 $\vdash A' \# t = t'$

with

A_i 's, A' , $\text{supp}(t)$, $\text{supp}(t') \subseteq A$

$\mathbb{T}_\lambda = (\Sigma_\lambda, E_\lambda)$

$\Sigma_\lambda(0) = \{ \langle a \rangle \mid a \in \mathbb{A} \} \cong \mathbb{A}$

$\Sigma_\lambda(1) = \{ \lambda_a \mid a \in \mathbb{A} \} \cong \mathbb{A}$

$\Sigma_\lambda(2) = \{ @ \} \cong 1$

$a \triangleright x \vdash a \# \lambda_a(x) \quad (\alpha)$

$a \triangleright a \# x \vdash \lambda_a(x @ \langle a \rangle) = x \quad (\eta)$

$a \triangleright a \# x, y \vdash \lambda_a(x) @ y = x \quad (\beta-1)$

\vdots

Nominal Equational Logic (Clouston & Pitts '07)

A NEL-theory $\mathbb{T} = (\Sigma, E)$

$\Sigma = \{ \Sigma(n) \in \mathbf{Nom} \}_{n \in \mathbb{N}}$

$t ::= \sigma x$
| $\circ t_1 \dots t_n$

where $\sigma \in \text{Perm}(\mathbb{A})$, $x \in \mathbf{V}$,
 $\circ \in \Sigma(n)$.

E is a set of axioms of the form

$A \triangleright A_1 \# x_1, \dots, A_n \# x_n$

$\vdash t = t'$

with

A_i 's, $\text{supp}(t), \text{supp}(t') \subseteq A$

$\mathbb{T}_\lambda = (\Sigma_\lambda, E_\lambda)$

$\Sigma_\lambda(0) = \{ \langle a \rangle \mid a \in \mathbb{A} \} \cong \mathbb{A}$

$\Sigma_\lambda(1) = \{ \lambda_a \mid a \in \mathbb{A} \} \cong \mathbb{A}$

$\Sigma_\lambda(2) = \{ @ \} \cong 1$

$a, b \triangleright b \# x \vdash \lambda_a(x) = \lambda_b((a b)x) \quad (\alpha)$

$a \triangleright a \# x \vdash \lambda_a(x @ \langle a \rangle) = x \quad (\eta)$

$a \triangleright a \# x, y \vdash \lambda_a(x) @ y = x \quad (\beta-1)$

\vdots

Nominal Equational Logic (Clouston & Pitts '07)

\mathbb{T} -Algebra (M, e)

$M \in \mathbf{Nom}$

$e_n : \Sigma(n) \times M^n \rightarrow M$

satisfying all axioms

$A \blacktriangleright \{A_i \# x_i\}_{1 \leq i \leq n}$
 $\vdash t = t'$

i.e.,

for all $\rho : \{x_1, \dots, x_n\} \rightarrow M$

s.t. $A_i \# \rho(x_i)$

$\bar{e}(t, \rho) = \bar{e}(t, \rho)$

\mathbb{T}_λ -Algebra (M, e)

$M \in \mathbf{Nom}$

$e_0 : \{ \langle a \rangle \mid a \in \mathbb{A} \} \times M^0 \rightarrow M$

$e_1 : \{ \lambda_a \mid a \in \mathbb{A} \} \times M^1 \rightarrow M$

$e_2 : \{ @ \} \times M^2 \rightarrow M$

For the axiom (α) ,
satisfying

$a, b \blacktriangleright b \# x$

$\vdash \lambda_a(x) = \lambda_b((a b)x)$

i.e.,

for all $\rho : \{x\} \rightarrow M$ s.t. $b \# \rho(x)$

$e_1(\lambda_a, \rho(x))$

$= e_1(\lambda_b, (a b) \cdot \rho(x))$

Nominal Equational Logic (Clouston & Pitts '07)

\mathbb{T} -Algebra (M, e)

$M \in \mathbf{Nom}$

$e_n : \Sigma(n) \times M^n \rightarrow M$

satisfying for all $\alpha : A \rightsquigarrow \mathbb{A}$

$\alpha(A) \blacktriangleright \{ \alpha(A_i) \# x_i \}_{1 \leq i \leq n}$
 $\vdash \alpha(t) = \alpha(t')$

Axioms are Invariant
under Renaming of atoms
by EQUIVARIANCE of e_n 's

for all $\alpha : A \rightsquigarrow \mathbb{A}$ and

for all $\rho : \{x_1, \dots, x_n\} \rightarrow M$

s.t. $\alpha(A_i) \# \rho(x_i)$

$\bar{e}(\alpha(t), \rho) = \bar{e}(\alpha(t'), \rho)$

\mathbb{T}_λ -Algebra (M, e)

$M \in \mathbf{Nom}$

$e_0 : \{ \langle a \rangle \mid a \in \mathbb{A} \} \times M^0 \rightarrow M$

$e_1 : \{ \lambda_a \mid a \in \mathbb{A} \} \times M^1 \rightarrow M$

$e_2 : \{ @ \} \times M^2 \rightarrow M$

For the axiom (α) ,

satisfying for all $\alpha : \{a, b\} \rightsquigarrow \mathbb{A}$

$\alpha(a), \alpha(b) \blacktriangleright \alpha(b) \# x$

$\vdash \lambda_{\alpha(a)}(x) = \lambda_{\alpha(b)}((\alpha(a) \alpha(b))x)$

i.e.,

for all $\alpha : \{a, b\} \rightsquigarrow \mathbb{A}$ and

for all $\rho : \{x\} \rightarrow M$ s.t. $\alpha(b) \# \rho(x)$

$e_1(\lambda_{\alpha(a)}, \rho(x))$

$= e_1(\lambda_{\alpha(b)}, (\alpha(a) \alpha(b)) \cdot \rho(x))$

NEL-Theories as Equational Systems

The Equational System $\overline{\mathbb{T}}$
for a NEL-theory $\mathbb{T} = (\Sigma, E)$

$$\overline{\mathbb{T}} = \mathbf{Nom} : \overline{\Sigma} \triangleright \overline{\Gamma} \vdash \overline{L} = \overline{R}$$

$$\overline{\Sigma} = \coprod_{n \in \mathbb{N}} \Sigma(n) \times (-)^n$$

$$\overline{\Gamma} = \coprod_{(\nabla \vdash t_1 = t_2) \in E} \Gamma_{\nabla}$$

$$\overline{L} = [F_{\nabla \vdash t_1}]_{(\nabla \vdash t_1 = t_2) \in E}$$

$$\overline{R} = [F_{\nabla \vdash t_2}]_{(\nabla \vdash t_1 = t_2) \in E}$$

$$\begin{aligned} \Gamma_{A \blacktriangleright \{A_i \# x_i\}}(M) \\ = \{ \alpha : A \rightarrow \mathbb{A}, \rho : \{x_i\} \rightarrow M \\ \mid \alpha(A_i) \# \rho(x_i) \} \end{aligned}$$

$$\begin{aligned} F_{A \blacktriangleright \{A_i \# x_i\} \vdash t}(M, e)(\alpha, \rho) \\ = \bar{e}(\alpha(t), \rho) \in M \end{aligned}$$

$$\begin{aligned} \overline{\Sigma}_{\lambda}(M) &= \{ \langle a \rangle \mid a \in \mathbb{A} \} \times M^0 \\ &+ \{ \lambda_a \mid a \in \mathbb{A} \} \times M^1 \\ &+ \{ @ \} \times M^2 \end{aligned}$$

For the axiom (α)

$$a, b \blacktriangleright b \# x \vdash \lambda_a(x) = \lambda_b((a b)x)$$

$$\begin{aligned} \Gamma_{a, b \blacktriangleright b \# x}(M) \\ = \{ \alpha : \{a, b\} \rightarrow \mathbb{A}, \rho : \{x\} \rightarrow M \\ \mid \alpha(b) \# \rho(x) \} \end{aligned}$$

$$\begin{aligned} F_{a, b \blacktriangleright b \# x \vdash \lambda_a(x)}(M, e)(\alpha, \rho) \\ = e_1(\lambda_{\alpha(a)}, \rho(x)) \in M \end{aligned}$$

$$\begin{aligned} F_{a, b \blacktriangleright b \# x \vdash \lambda_b((a b)x)}(M, e)(\alpha, \rho) \\ = e_1(\lambda_{\alpha(b)}, (\alpha(a) \alpha(b)) \cdot \rho(x)) \in M \end{aligned}$$

Properties of Categories of Algebras for NEL-Theories

$\bar{\Sigma}, \bar{\Gamma}$ preserve filtered colimits and epimorphisms. (Easy!)

$$\begin{array}{ccc} & \mathbb{T}\text{-Alg} & \\ & \uparrow & \searrow \\ F_{\mathbb{T}} & \dashv & U_{\mathbb{T}} \\ & \downarrow & \\ & \mathbf{Nom} & \end{array}$$

- $\bar{\mathbb{T}}\text{-Alg}$ is monadic over \mathbf{Nom} .
- $\bar{\mathbb{T}}\text{-Alg}$ is cocomplete.

Outline of Completeness Proof

For a NEL-theory \mathbb{T}

$$\text{(Goal)} \quad \nabla \vDash t = t' \implies \nabla \vdash t = t'$$

- Define a nominal set of terms \mathbf{Tm} with syntactic structure e , a set of rules \mathcal{R}_\approx generating \approx on \mathbf{Tm} s.t. $(\mathbf{Tm}/\approx, e/\approx)$ is a \mathbb{T} -algebra.

- $\nabla \vDash t = t'$
 - $\Rightarrow \nabla \vDash_{(\mathbf{Tm}/\approx, e/\approx)} t = t'$
 - $\Rightarrow [t]_\approx = [t']_\approx \in \mathbf{Tm}/\approx$
 - $\Rightarrow t \approx t'$ is provable by \mathcal{R}_\approx
 - $\Rightarrow \nabla \vdash t = t'$ (by simulating \mathcal{R}_\approx in \mathbb{T})

-

Outline of Completeness Proof

For a NEL-theory \mathbb{T}

$$\text{(Goal)} \quad \nabla \vDash t = t' \implies \nabla \vdash t = t'$$

- Define a nominal set of terms \mathbf{Tm} with syntactic structure e , a set of rules \mathcal{R}_\approx generating \approx on \mathbf{Tm} s.t. $(\mathbf{Tm}/\approx, e/\approx)$ is a \mathbb{T} -algebra.

- $\nabla \vDash t = t'$
 - $\implies \nabla \vDash_{(\mathbf{Tm}/\approx, e/\approx)} t = t'$
 - $\implies [\text{enc}(\nabla \vdash t)]_\approx = [\text{enc}(\nabla \vdash t')]_\approx \in \mathbf{Tm}/\approx$
 - $\implies \text{enc}(\nabla \vdash t) \approx \text{enc}(\nabla \vdash t')$ is provable by \mathcal{R}_\approx
 - $\implies \nabla \vdash t = t'$ (by simulating \mathcal{R}_\approx in \mathbb{T})

-

Outline of Completeness Proof

For a NEL-theory \mathbb{T}

$$\text{(Goal)} \quad \nabla \vDash t = t' \implies \nabla \vdash t = t'$$

- Define a nominal set of terms \mathbf{Tm} with syntactic structure e , a set of rules \mathcal{R}_\approx generating \approx on \mathbf{Tm} s.t. $(\mathbf{Tm}/\approx, e/\approx)$ is a \mathbb{T} -algebra.
- $\nabla \vDash t = t'$
 - $\implies \nabla \vDash_{(\mathbf{Tm}/\approx, e/\approx)} t = t'$
 - $\implies [\text{enc}(\nabla \vdash t)]_\approx = [\text{enc}(\nabla \vdash t')]_\approx \in \mathbf{Tm}/\approx$
 - $\implies \text{enc}(\nabla \vdash t) \approx \text{enc}(\nabla \vdash t')$ is provable by \mathcal{R}_\approx
 - $\implies \nabla \vdash t = t'$ (by simulating \mathcal{R}_\approx in \mathbb{T})
- $A \blacktriangleright \{A_i \# x_i\} \vdash x_i \xrightarrow{\text{enc}} x_i^{\text{sort}(\overline{A-A_i})} \in \mathbf{Tm}$

Outline of Completeness Proof

For a NEL-theory \mathbb{T}

$$\text{(Goal)} \quad \nabla \vDash t = t' \implies \nabla \vdash t = t'$$

- Define a nominal set of terms \mathbf{Tm} with syntactic structure e , a set of rules \mathcal{R}_\approx generating \approx on \mathbf{Tm} s.t. $(\mathbf{Tm}/\approx, e/\approx)$ is a \mathbb{T} -algebra.

- $\nabla \vDash t = t'$
 $\implies \nabla \vDash_{(\mathbf{Tm}/\approx, e/\approx)} t = t'$
 $\implies [\text{enc}(\nabla \vdash t)]_\approx = [\text{enc}(\nabla \vdash t')]_\approx \in \mathbf{Tm}/\approx$
 $\implies \text{enc}(\nabla \vdash t) \approx \text{enc}(\nabla \vdash t')$ is provable by \mathcal{R}_\approx
 $\implies \nabla \vdash t = t'$ (by simulating \mathcal{R}_\approx in \mathbb{T})

- $A \blacktriangleright \{A_i \# x_i\} \vdash t \xrightarrow{\text{enc}} t\{x_i^{\sigma \cdot \text{sort}(\overline{A - A_i})} / \sigma x_i\} \in \mathbf{Tm}$
e.g. $a, b \blacktriangleright b \# x \vdash \lambda_a(x) = \lambda_b((a b)x) \xrightarrow{\text{enc}} \lambda_a(x^a) = \lambda_b(x^b)$

Construction of Term-Algebra

For a NEL-theory $\mathbb{T} = (\Sigma, E)$

For a countable set of variables V ,

$$\tilde{V} \triangleq V \times \text{DList}_{\text{fin}}(\mathbb{A}) = \{x^{\vec{A}} \mid x \in V, \vec{A} \in \text{DList}_{\text{fin}}(\mathbb{A})\}$$

From the construction

$$0 \rightarrow \tilde{V} + \bar{\Sigma}0 \rightarrow \tilde{V} + \bar{\Sigma}(\tilde{V} + \bar{\Sigma}0) \rightarrow \dots \rightarrow T(\tilde{V})$$

A Free $\bar{\Sigma}$ -algebra $(T(\tilde{V}), e)$ on \tilde{V}

$$t \in T(\tilde{V}) \quad ::= \quad x^{\vec{A}} \quad (x \in V, \vec{A} \in \text{DList}_{\text{fin}}(\mathbb{A}))$$
$$\quad \quad \quad | \quad \circ t_1 \dots t_n \quad (\circ \in \Sigma(n))$$

$$e_n(\circ, t_1, \dots, t_n) \quad ::= \quad \circ t_1 \dots t_n \quad (\circ \in \Sigma(n))$$

Construction of Term-Algebra

From the construction

$$\begin{array}{ccccccc}
 \bar{\Sigma}T(\tilde{V}) & \xrightarrow{\bar{\Sigma}c_1} & \bar{\Sigma}T(\tilde{V})_1 & \xrightarrow{\bar{\Sigma}c_2} & \bar{\Sigma}T(\tilde{V})_2 & \cdots & \bar{\Sigma}T(\tilde{V})/\approx \\
 \downarrow e & \searrow e_1 & \text{po} & \searrow e_2 & \text{po} & \searrow e_3 & \downarrow \exists! e/\approx \\
 T(\tilde{V}) & \xrightarrow[\text{coeq}]{c_1} & T(\tilde{V})_1 & \xrightarrow{c_2} & T(\tilde{V})_2 & \xrightarrow{c_3} & T(\tilde{V})_3 \cdots T(\tilde{V})/\approx \\
 \bar{L}(T(\tilde{V}),e) \uparrow \uparrow \bar{R}(T(\tilde{V}),e) & & & & & & \bar{L}(T(\tilde{V})/\approx,e/\approx) \stackrel{\Delta}{=} \bar{R}(T(\tilde{V})/\approx,e/\approx) \\
 \bar{\Gamma}T(\tilde{V}) & \xrightarrow{\hspace{15em}} & & & & & \bar{\Gamma}T(\tilde{V})/\approx
 \end{array}$$

Construction of Term-Algebra

From the construction

$$\begin{array}{ccccccc}
 \bar{\Sigma}T(\tilde{V}) & \xrightarrow{\bar{\Sigma}c_1} & \bar{\Sigma}T(\tilde{V})_1 & \xrightarrow{\bar{\Sigma}c_2} & \bar{\Sigma}T(\tilde{V})_2 & \cdots & \bar{\Sigma}T(\tilde{V})/\approx \\
 \downarrow e & \searrow e_1 & \text{po} & \searrow e_2 & \text{po} & \searrow e_3 & \downarrow \exists! e/\approx \\
 T(\tilde{V}) & \xrightarrow[\text{coeq}]{c_1} & T(\tilde{V})_1 & \xrightarrow{c_2} & T(\tilde{V})_2 & \xrightarrow{c_3} & T(\tilde{V})_3 \cdots T(\tilde{V})/\approx \\
 \bar{L}(T(\tilde{V}),e) \uparrow \uparrow \bar{R}(T(\tilde{V}),e) & & & & & & \bar{L}(T(\tilde{V})/\approx,e/\approx) \stackrel{\Delta}{=} \bar{R}(T(\tilde{V})/\approx,e/\approx) \\
 \bar{\Gamma}T(\tilde{V}) & \xrightarrow{\hspace{15em}} & & & & & \bar{\Gamma}T(\tilde{V})/\approx
 \end{array}$$

Rules \mathcal{R}_{\approx}

$$\begin{array}{l}
 \text{Ax} \frac{\alpha : A \rightarrow A, \{s_i \in T(\tilde{V}) \mid \alpha(A_i) \# \text{supp}(s_i)\}}{\alpha(t)\{\sigma \cdot s_i/\sigma x_i\} \approx \alpha(t')\{\sigma \cdot s_i/\sigma x_i\}} \quad (A \blacktriangleright \{A_i \# x_i\} \vdash t = t') \in E \\
 \\
 \text{Cong} \frac{\forall i \ s_i \approx s'_i}{o \ s_1 \dots s_n \approx o \ s'_1 \dots s'_n} \quad o \in \Sigma(n) \qquad \text{Refl} \qquad \text{Symm} \qquad \text{Trans}
 \end{array}$$

A Free $\bar{\mathbb{T}}$ -algebra $(T(\tilde{V})/\approx, e/\approx)$ on $(T(\tilde{V}), e)$

$$e/\approx(o, [t_1]_{\approx}, \dots, [t_n]_{\approx}) ::= [o \ t_1 \dots t_n]_{\approx} \quad (o \in \Sigma(n))$$

Completeness of Nominal Equational Logic

Recall $\text{enc}(A \blacktriangleright \{A_i \# x_i\} \vdash t) \triangleq t[x_i^{\sigma \cdot \text{sort}(\overrightarrow{A-A_i})} / \sigma x_i]$

$$A \blacktriangleright \{A_i \# x_i\} \models t = t'$$

$$\Rightarrow A \blacktriangleright \{A_i \# x_i\} \models_{(T(\tilde{V})/\approx, e/\approx)} t = t'$$

$$\Rightarrow \overline{e/\approx}(t, \rho : x_i \mapsto [x_i^{\text{sort}(\overrightarrow{A-A_i})}])_{\approx} = \overline{e/\approx}(t', \rho : x_i \mapsto [x_i^{\text{sort}(\overrightarrow{A-A_i})}])_{\approx}$$

$$\Rightarrow [t\{x_i^{\sigma \cdot \text{sort}(\overrightarrow{A-A_i})} / \sigma x_i\}]_{\approx} = [t'\{x_i^{\sigma \cdot \text{sort}(\overrightarrow{A-A_i})} / \sigma x_i\}]_{\approx}$$

$$\Rightarrow \text{enc}(A \blacktriangleright \{A_i \# x_i\} \vdash t) \approx \text{enc}(A \blacktriangleright \{A_i \# x_i\} \vdash t')$$

By induction of the size of proof trees,

$$\begin{aligned} \text{enc}(\nabla \vdash t) \approx \text{enc}(\nabla \vdash t') &\text{ provable by } \mathcal{R}_{\approx} \\ \Rightarrow \nabla \vdash t = t' &\text{ provable in } \mathbb{T} \end{aligned}$$

Simplification of NEL Proofs

Rules \mathcal{R}_{\top}

$$\begin{array}{c}
 \text{REFL} \frac{}{\nabla \vdash t = t} \quad \text{SYMM} \frac{\nabla \vdash A \# t = t'}{\nabla \vdash A \# t' = t} \quad \text{TRANS} \frac{\nabla \vdash A_1 \# t = t' \quad \nabla \vdash A_2 \# t' = t''}{\nabla \vdash (A_1 \cup A_2) \# t = t''} \\
 \text{SUBST} \frac{\nabla' \vdash \sigma = \sigma' : \nabla \quad \nabla \vdash A \# t = t'}{\nabla' \vdash A \# t\{\sigma\} = t'\{\sigma'\}} \quad \sigma, \sigma' \text{ substitutions} \\
 \text{WEAK} \frac{\nabla \vdash A \# t = t'}{\nabla' \vdash A \# t = t'} \quad \nabla \leq \nabla' \quad \text{ATM-INTRO} \frac{\nabla \vdash A \# t = t'}{\nabla \#^a \vdash A \cup \{a\} \# t = t'} \quad a \# (A, t, t') \\
 \text{ATM-ELIM} \frac{\nabla \#^a \vdash A \# t = t'}{\nabla \vdash A \cup \{a\} \# t = t'} \quad a \# (\nabla, A, t, t') \quad \text{\#-EQUIVAR} \frac{}{A \blacktriangleright A_1 \# x \vdash \pi \cdot A_1 \# \pi x} \\
 \text{SUSP} \frac{}{A \blacktriangleright \{a \mid \pi(a) \neq \pi'(a)\} \# x \vdash \pi \cdot x \# \pi' x} \quad \text{AXIOM} \frac{}{\nabla \vdash A \# t = t'} \quad (\nabla \vdash A \# t = t') \in E
 \end{array}$$

Rules \mathcal{R}_{\approx}

$$\begin{array}{c}
 \text{Ax} \frac{\alpha : A \mapsto A, \{s_i \mid \alpha(A_i) \# \text{supp}(s_i)\}}{\alpha(t)\{\sigma \cdot s_i / \sigma x_i\} \approx \alpha(t')\{\sigma \cdot s_i / \sigma x_i\}} \quad (A \blacktriangleright \{A_i \# x_i\} \vdash t = t') \in E \\
 \text{Cong} \frac{\forall i \ s_i \approx s'_i}{o \ s_1 \dots s_n \approx o \ s'_1 \dots s'_n} \quad o \in \Sigma(n) \quad \text{RefI} \quad \text{Symm} \quad \text{Trans}
 \end{array}$$

$\nabla \vdash t = t'$ provable by \mathcal{R}_{\top}

$\Leftrightarrow \text{enc}(\nabla \vdash t) \approx \text{enc}(\nabla \vdash t')$ provable by \mathcal{R}_{\approx}

$\Leftrightarrow \text{enc}(\nabla \vdash t) \xrightarrow{\text{Ax; Cong}^*} s_1 \quad \dots \quad s_{n-1} \xrightarrow{\text{Ax; Cong}^*} \text{enc}(\nabla \vdash t')$

① Advantages of Equational Systems

- More expressive than Enriched Algebraic Theories
- Free theorems about Categories of Algebras
- Explicit Construction of Free Algebras

② Advantages of Inequational Systems

- Semantic Models of Rewriting Systems.
- Free Algebras are Syntactic Term Models.
- Abstract Notions and Theorems can be developed.
 - Abstract Criteria for Modularity of Confluence, Termination

In future

- Abstract notion of Orthogonality.
- Abstract Semantic Labeling.

...

③ Nominal Equational Logic (using Equational Systems)

- Monadicity and Cocompleteness of Categories of Algebras
- Completeness of Nominal Equational Logic
- Simplification of Proofs