Equational Systems and Free Constructions

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Contributions of the paper

1. General abstract Definition of Equational System.
2. Development of the Theory of Equational Systems.
3. Applications of Equational Systems.

Overview: Definition of Equational System

Equational Systems are a framework for defining models of systems.

- **Models of Logics**
  what we want to reason about.
- **Models of Computational Calculi**
  semantic domains where meanings of programs are defined.
  e.g. λ-calculus, π-calculus, ...
- **Models of Data Types**
  semantic domains where data types and type constructors are interpreted.

⋮

Overview: Theory of Equational Systems

$$\mathsf{Model}(\mathbb{S}) \underset{F}{\overset{U}{\rightleftarrows}} \mathscr{D}, \qquad F \dashv U$$

1. **Construction of free models**

   Theoretically, the models of $\mathbb{S}$ can be represented by a monad.
   Practically, it gives interesting models:
   - it may give syntactic models. (initial algebra semantics)
   - it may give fully abstract models.

   ⋮

2. **Model($\mathbb{S}$) is cocomplete.**

   Models can be combined to form new ones (e.g. by coproducts or pushouts) in a compositional fashion.

Overview: Applications of Equational System

- Algebraic Theories
  - First-order equational logic
  - Specification, correctness and implementation of abstract data types [ADJ Group '78]

- Enriched Algebraic Theories [Kelly & Power '93]
  - Algebraic treatment of computational effects [Plotkin & Power '03, '04]

- Equational Systems
  - Σ-monoids [Fiore, Plotkin & Turi '99]
  - π-algebras [Stark '05]
  - Nominal equational logic [Clouston & Pitts '07]

Motivation for definition: I. Signatures

Signatures as Endofunctors

- **Algebraic Theory**

$\Sigma_{\mathsf{Num}} = \{ \mathsf{zero} : 0, \mathsf{succ} : 1, \mathsf{plus} : 2 \}$

Σ_{Num}-algebra

- $D \in \mathbf{Set}$
- $[\![\mathsf{zero}]\!] : D^0 \to D$
- $[\![\mathsf{succ}]\!] : D^1 \to D$
- $[\![\mathsf{plus}]\!] : D^2 \to D$

- **Equational System**

$\Sigma_{\mathsf{Num}}(X) = X^0 + X^1 + X^2$ on $\mathbf{Set}$

Σ_{Num}-algebra

- $D \in \mathbf{Set}$
- $s : \Sigma_{\mathsf{Num}} D \to D$
  $: D^0 + D^1 + D^2 \to D$

Motivation for definition: II. Equations

Equations as parallel pairs of Functors

$$\{\, x, y \,\} \vdash \mathsf{plus}(\mathsf{succ}(x), y) = \mathsf{succ}(\mathsf{plus}(x, y))$$

Algebraic Theory

$$\begin{array}{ccc} 1 & D & D^2 \\ {\scriptstyle [\![\mathsf{zero}]\!]}\downarrow & {\scriptstyle [\![\mathsf{succ}]\!]}\downarrow & {\scriptstyle [\![\mathsf{plus}]\!]}\downarrow \\ D & D & D \end{array} \longmapsto \quad \forall \rho : \{\, x, y \,\} \to D \quad [\![\mathsf{plus}(\mathsf{succ}(x), y)]\!]_\rho = [\![\mathsf{succ}(\mathsf{plus}(x, y))]\!]_\rho \in D$$

Equational System

$$\boxed{\Sigma_{\mathsf{Num}}\text{-}\mathbf{Alg}} \quad \rightrightarrows \quad \boxed{(-)^2\text{-}\mathbf{Alg}}$$

$$\begin{array}{c} 1 + D + D^2 \\ {\scriptstyle [[\![\mathsf{zero}]\!],[\![\mathsf{succ}]\!],[\![\mathsf{plus}]\!]]}\downarrow \\ D \end{array} \longmapsto \quad \begin{array}{ccc} & D^2 & \\ {\scriptstyle [\![\mathsf{succ}]\!] \times id}\swarrow & & \searrow{\scriptstyle [\![\mathsf{plus}]\!]} \\ D^2 & = & D \\ {\scriptstyle [\![\mathsf{plus}]\!]}\searrow & & \swarrow{\scriptstyle [\![\mathsf{succ}]\!]} \\ & D & \end{array}$$

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$$1 \xrightarrow{[\![\mathsf{zero}]\!]} D \qquad D \xrightarrow{[\![\mathsf{succ}]\!]} D \qquad D^2 \xrightarrow{[\![\mathsf{plus}]\!]} D \quad \longmapsto \quad \forall \rho : \{\, x, y \,\} \to D \quad [\![\mathsf{plus}(\mathsf{succ}(x), y)]\!]_\rho = [\![\mathsf{succ}(\mathsf{plus}(x, y))]\!]_\rho \in D$$

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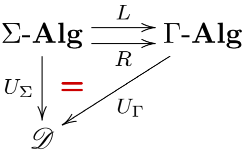
Definition of Equational System



- **Equational System** $\mathbb{T}$

$$(\mathscr{D} \rhd \Sigma \vdash L = R : \Gamma)$$

- **$\mathbb{T}$-Algebra**

$$(D, s : \Sigma D \to D)$$

such that

$$L(D, s) = R(D, s)$$

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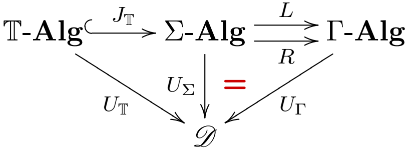

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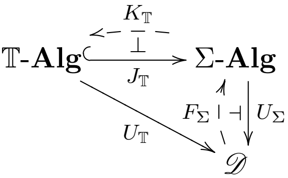
Theorem: Basic Free Construction

For $\mathbb{T} = (\mathscr{D} \rhd \Sigma \vdash L = R : \Gamma)$ an Equational System,



$\mathscr{D}$ is cocomplete.
Σ, Γ preserve ω-colimits.
(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \to V + \Sigma\,0 \to V + \Sigma\,(V + \Sigma 0) \to \cdots \to (V + \Sigma(\text{-}))^{*}0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \to X)$

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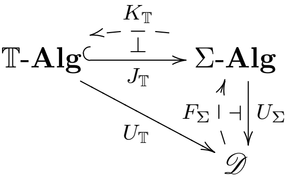

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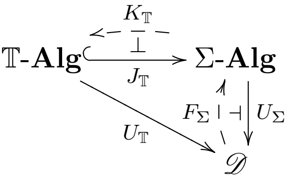

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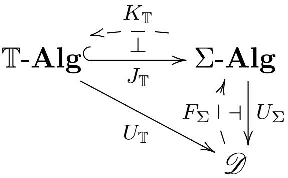

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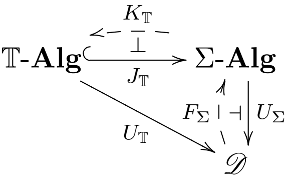

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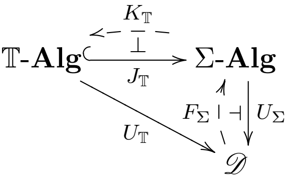

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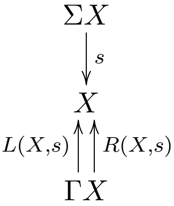

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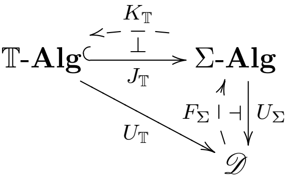

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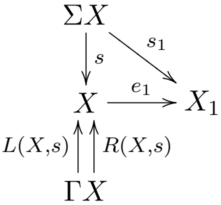

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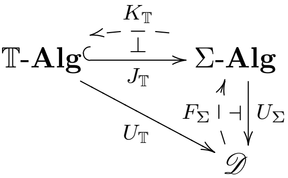

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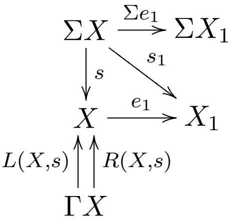

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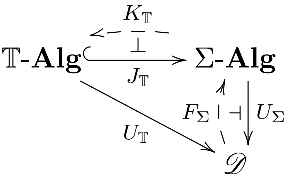

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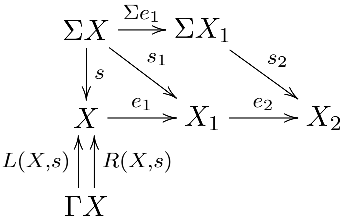

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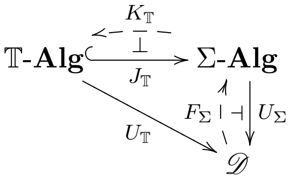

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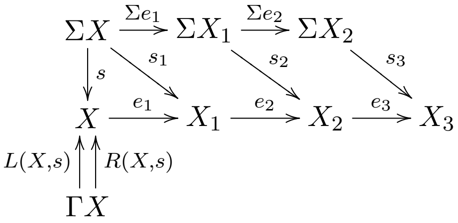

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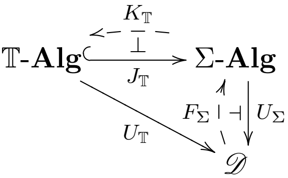

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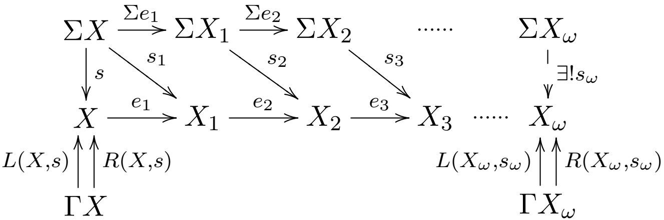

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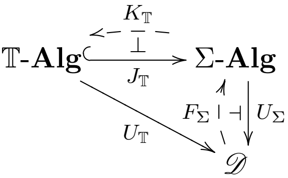

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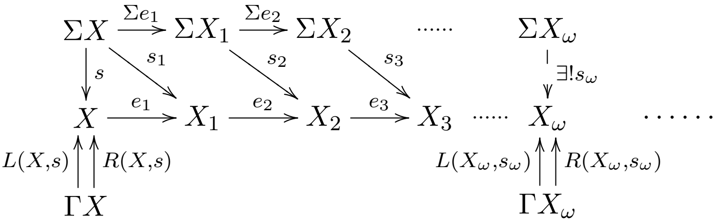

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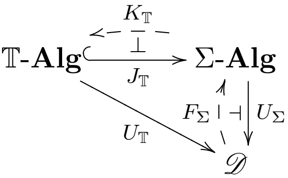

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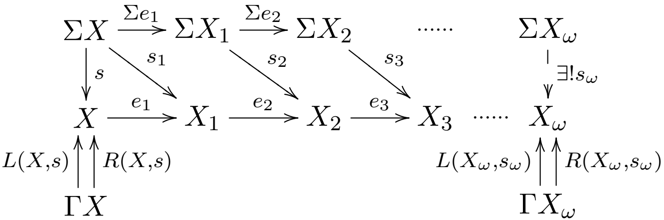
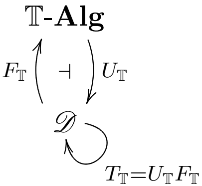


Theorem: Properties of Categories of Algebras

For $\mathbb{T} = (\mathscr{D} \rhd \Sigma \vdash L = R : \Gamma)$ an Equational System,



> $\mathscr{D}$ is cocomplete.
> Σ, Γ preserve ω-colimits.

1. $\mathbb{T}$-**Alg** is cocomplete.
2. $\mathbb{T}$-**Alg** is monadic over $\mathscr{D}$.
3. $T_{\mathbb{T}}$ preserves colimits of ω-chains.

An Application: Nominal Equational Logic (Clouston & Pitts '07)

$$\textbf{Term} = a, b, c, \ldots \mid \lambda[a]\,\textbf{Term} \mid \textbf{Term}@\textbf{Term}$$

- **A NEL-theory $\mathbb{T}_\lambda$ for λ-calculus**

$$\langle - \rangle : \mathbb{A} \to \mathsf{tm}$$
$$\lambda : \mathbb{A} \times \mathsf{tm} \to \mathsf{tm}$$
$$-@- : \mathsf{tm} \times \mathsf{tm} \to \mathsf{tm}$$

$$a : \mathbb{A}, x : \mathsf{tm} \vdash a \# \lambda(a, x) : \mathsf{tm} \qquad (\alpha)$$
$$a : \mathbb{A}, a \# x : \mathsf{tm} \vdash \lambda(a, x@\langle a \rangle) = x : \mathsf{tm} \qquad (\eta)$$
$$a : \mathbb{A}, a \# x : \mathsf{tm}, y : \mathsf{tm} \vdash \lambda(a, x)@y = x : \mathsf{tm} \qquad (\beta\text{-}1)$$
$$\vdots$$

- **An Equational System for $\mathbb{T}_\lambda$**

$$(\mathbf{Nom} \rhd \Sigma_\lambda \vdash L_\lambda = R_\lambda : \Gamma_\lambda)$$
$$\Sigma_\lambda D = \mathbb{A} + \mathbb{A} \times D + D \times D$$
$$\Gamma_\lambda D = \mathbb{A} \times D + \mathbb{A} \otimes D + (\mathbb{A} \otimes D) \times D + \ldots$$

Concluding Remarks

Conclusion

- As an advantage, Equational Systems provide a general, abstract and practical theory for the specification and free construction of equational models suitable for modern applications.
- As a drawback, but in favour of generality, Equational Systems do not have associated syntactic (Lawvere) theories in general.

Further work

- Enriched Equational Systems
- Equational Cosystems
- Inequational Systems – Rewriting
    - Modularity of Confluence
    - Confluence of Orthogonal Systems
- Conditional (In)equational System