Term Equational Systems and Logics

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Outline of the talk

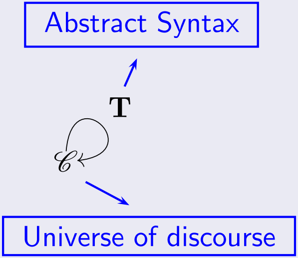
1. Definition of Term Equational System (TES)
2. Theory of Term Equational Systems & Logics
3. Applications

Overview

Overview: Definition of TES

Term Equational Systems are a framework for developing equational logics.

Idea of Term Equational System



$\Longrightarrow$

- Terms in context
- Equational judgements
- Algebraic model theory

Overview: Theory of TES & TEL

1. Term Equational Logic (TEL)
   - Deductive system
   - Sound (not necessarily complete)
2. Internal completeness & Construction of free models
   - Reasoning by rewriting
   - Sound and Complete

Overview: Applications

1. (Multi-sorted) First-order equational logic [treated in the talk]
2. (Multi-sorted) Nominal equational logic [treated in the paper]
   (Gabbay & Mathijssen 06; Clouston & Pitts 07)
3. (Multi-sorted) Binding equational logic
   (Hamana 03)
4. (Multi-sorted) Second-order equational logic
   in the context of second-order abstract syntax (Fiore 08)

   (cf. Equational fragment of Combinatory Reduction System (Klop 80))

Definition

Definition: I. Enriched universe and Monad

- Enriched universe $\mathcal{U} = (\mathscr{V}, \underline{\mathscr{C}})$
  - $\mathscr{V}$: a symmetric monoidal closed category
  - $\underline{\mathscr{C}}$: a $\mathscr{V}$-category with tensors $\otimes$ and powers $[-, =]$

$$- \otimes = : \mathscr{V} \times \mathscr{C} \to \mathscr{C} \qquad [-, =] : \mathscr{V} \times \mathscr{C} \to \mathscr{C}$$

- $\mathscr{V}$-monad $\underline{\mathbf{T}}$ on $\underline{\mathscr{C}}$
  $\cong$ monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathscr{C}$ with strength τ

$$\tau_{V,X} : V \otimes TX \to T(V \otimes X)$$

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Main example of enriched universe

Symmetric monoidal closed category (enriched over itself)

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$$\tau_{V,X} : V \otimes TX \to T(V \otimes X)$$

Example (Multi-sorted First-order Equational Logic)

For S a set of sorts, Σ an S-sorted signature,

- $\mathcal{U}_\Sigma = (\boldsymbol{\mathcal{S}et}, \boldsymbol{\mathcal{S}et}^{\mathsf{S}})$

$$\begin{aligned} V \otimes \{X_{\mathsf{s}}\}_{\mathsf{s} \in \mathsf{S}} &= \{V \times X_{\mathsf{s}}\}_{\mathsf{s} \in \mathsf{S}} \\ [V, \{X_{\mathsf{s}}\}_{\mathsf{s} \in \mathsf{S}}] &= \{[V, X_{\mathsf{s}}]\}_{\mathsf{s} \in \mathsf{S}} \end{aligned}$$

- $\mathbf{T}_\Sigma$ with strength $\overline{\tau}$ = free monad on the endofunctor F_Σ with strength $\tau_{V,X} : V \otimes F_\Sigma X \to F_\Sigma(V \otimes X)$

$$F_\Sigma(\{X_{\mathsf{s}}\}_{\mathsf{s} \in \mathsf{S}}) = \Big\{ \coprod_{f : \mathsf{s}_1 \ldots \mathsf{s}_n \to \mathsf{s} \in \Sigma} X_{\mathsf{s}_1} \times \ldots \times X_{\mathsf{s}_n} \Big\}_{\mathsf{s} \in \mathsf{S}}$$

$$\tau_{V,X}(v, f(x_1, \ldots, x_n)) = f((v, x_1), \ldots, (v, x_n))$$

Definition: II. Terms and Judgements

Definition (Terms and Judgements)

Given $\mathcal{U} = (\mathscr{V}, \underline{\mathscr{C}})$ and $\underline{\mathbf{T}} = (T, \eta, \mu, \tau)$,

1. Generalised Term $A, C \vdash t$ as $t : A \to TC \in \mathscr{C}$
2. Generalised Equation $A, C \vdash t \equiv t'$

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Example (continued)

Syntactic terms $\cong$ Generalised terms

$$\{\, t \mid \Gamma \vdash t : \mathsf{s} \,\} \;\cong\; T_\Sigma \widetilde{\Gamma}(\mathsf{s}) \;\cong\; \boldsymbol{\mathcal{S}et}^{\mathsf{S}}(\widetilde{\mathsf{s}}, T_\Sigma \widetilde{\Gamma})$$

where

$$\widetilde{\mathsf{s}}(\mathsf{r}) = \begin{cases} 1 & \text{if } \mathsf{r} = \mathsf{s} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\widetilde{\Gamma}(\mathsf{r}) = \{\, x \mid x : \mathsf{r} \in \Gamma \,\}$$

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Object Variables | Meta Variables | in higher order examples

Example (continued)

Syntactic terms ≅ Generalised terms

$$\{\, t \mid \Gamma \vdash t : \mathsf{s} \,\} \;\cong\; T_\Sigma\widetilde{\Gamma}(\mathsf{s}) \;\cong\; \boldsymbol{Set}^{\mathsf{S}}(\widetilde{\mathsf{s}}, T_\Sigma\widetilde{\Gamma})$$

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Definition (Models and Satisfaction relation)

Given $\mathcal{U} = (\mathscr{V}, \underline{\mathscr{C}})$ and $\underline{\mathbf{T}} = (T, \eta, \mu, \tau)$,

- Models for $\mathbf{T}$: Eilenberg-Moore algebras for $\mathbf{T}$

$$(X, \xi : TX \to X)$$

- Satisfaction relation

$$(X, \xi) \models A, C \vdash t \equiv t' \iff \underline{\mathscr{C}}(C, X) \otimes A \underset{[\![t']\!]_{(X,\xi)}}{\overset{[\![t]\!]_{(X,\xi)}}{\rightrightarrows}} X \quad (\text{equal})$$

where $[\![t]\!]_{(X,\xi)}$ is the transpose of

$$\underline{\mathscr{C}}(C, X) \xrightarrow{\mathbf{T}} \underline{\mathscr{C}}(\mathbf{T}C, \mathbf{T}X) \xrightarrow{\underline{\mathscr{C}}(t,\xi)} \underline{\mathscr{C}}(A, X)$$

Definition: III. Model Theory

Definition (Models and Satisfaction relation)

Given $\mathcal{U} = (\mathscr{V}, \underline{\mathscr{C}})$ and $\underline{\mathbf{T}} = (T, \eta, \mu, \tau)$,

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Example (continued)

- Models for $\mathbf{T}_\Sigma$: F_Σ-algebras $(X, \xi : F_\Sigma X \to X)$

$$\forall_{f : \mathsf{s}_1 \ldots \mathsf{s}_n \to \mathsf{s} \in \Sigma} \quad \xi_f : X(\mathsf{s}_1) \times \ldots \times X(\mathsf{s}_n) \to X(\mathsf{s})$$

- For $(\Gamma \vdash t \equiv t' : \mathsf{s})$,

$$[\![t]\!] : \quad \prod_{\mathsf{r} \in \mathsf{S}} X(\mathsf{r})^{\{x \mid x : \mathsf{r} \in \Gamma\}} \to X(\mathsf{s})$$

Definition: IV. Term Equational System

Definition (Term Equational System)

- A TES $\mathbb{S} = (\mathscr{V}, \underline{\mathscr{C}}, \underline{\mathbf{T}}, \mathcal{A})$ is given by
  - a enriched universe $(\mathscr{V}, \underline{\mathscr{C}})$,
  - a $\mathscr{V}$-monad $\underline{\mathbf{T}}$, and
  - a set $\mathcal{A}$ of generalised equations.
- Models for $\mathbb{S}$ ($\mathbb{S}$-algebras): Models for $\mathbf{T}$ satisfying $\mathcal{A}$
- $\mathbb{S}$-$\mathbf{Alg}$: The category of $\mathbb{S}$-algebras.

Theory I.
Term Equational Logic

Theory: Term Equational Logic

- Equivalence relation

$$\frac{}{A, C \vdash t \equiv t} \qquad \frac{A, C \vdash t \equiv t'}{A, C \vdash t' \equiv t} \qquad \frac{A, C \vdash t \equiv t' \quad A, C \vdash t' \equiv t''}{A, C \vdash t \equiv t''}$$

- Axiom

$$\frac{}{A, C \vdash t \equiv t'} \, (A, C \vdash t \equiv t') \in \mathcal{A}$$

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Example: Multi-sorted First-order Equational Logic (MFoEL)

$$\frac{}{\Gamma \vdash t \equiv t' : \mathsf{s}} \qquad \frac{\Gamma \vdash t \equiv t' : \mathsf{s}}{\Gamma \vdash t' \equiv t : \mathsf{s}} \qquad \frac{\Gamma \vdash t \equiv t' : \mathsf{s} \quad \Gamma \vdash t' \equiv t'' : \mathsf{s}}{\Gamma \vdash t \equiv t'' : \mathsf{s}}$$

$$\text{Axiom} \, \frac{}{\Gamma \vdash t \equiv t' : \mathsf{s}} \, (\Gamma \vdash t \equiv t' : \mathsf{s}) \in \mathcal{A}$$

Theory: Term Equational Logic

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- Axiom

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- Local Character

$$\frac{A_i, C \vdash t \circ e_i \equiv t' \circ e_i \quad (i \in I)}{A, C \vdash t \equiv t'} \; \{ e_i : A_i \to A \}_{i \in I} \text{ jointly epi}$$

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- Substitution

$$\frac{A, B \vdash t \equiv t' \quad B, C \vdash s \equiv s'}{A, C \vdash t[s] \equiv t'[s']} \longrightarrow A \xrightarrow{t} TB \xrightarrow{Ts} TTC \xrightarrow{\mu} TC$$

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$$\cfrac{\{x_i : \mathsf{s}_i\}_i \vdash t \equiv t' : \mathsf{t} \qquad \cfrac{\{ \Gamma \vdash s_i \equiv s'_i : \mathsf{s}_i \}_{i \in I}}{\Gamma \vdash \{s_i\}_i \equiv \{s'_i\}_i : \{\mathsf{s}_i\}_i} \text{(by Loc. Char.)}}{\Gamma \vdash t[x_i \mapsto s_i]_i \equiv t'[x_i \mapsto s'_i]_i : \mathsf{t}} \text{(by Substitution)}$$

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Subst ← Loc. Char. + Substitution

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- Tensor Extension

$$\frac{A, C \vdash t \equiv t'}{V \otimes A, V \otimes C \vdash \langle V \rangle t \equiv \langle V \rangle t'}$$

$$V \otimes A \xrightarrow{id \otimes t} V \otimes TC \xrightarrow{\tau} T(V \otimes C)$$

Theory: Soundness of TEL

Theorem (Soundness of TEL)

For a TES $\mathbb{S} = (\mathscr{V}, \underline{\mathscr{C}}, \underline{\mathbf{T}}, \mathcal{A})$

$$A, C \vdash t \equiv t' \quad \textit{is derivable from } \mathcal{A}$$

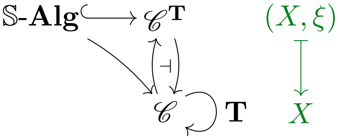
$$\Downarrow$$

$$\forall\, (X, \xi) \in \mathbb{S}\text{-}\mathbf{Alg} \qquad (X, \xi) \;\models\; A, C \vdash t \equiv t'$$

Theory II.
Internal Completeness & Free Construction

Theory: Internal Completeness

For a TES $\mathbb{S} = (\mathscr{V}, \underline{\mathscr{C}}, \underline{\mathbf{T}}, \mathcal{A})$



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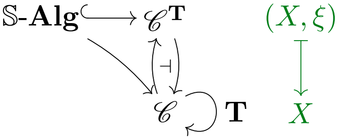

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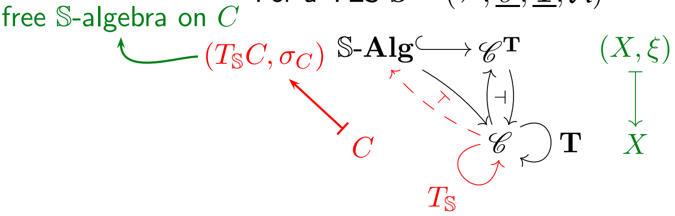

$$\forall\,(X,\xi) \in \mathbb{S}\text{-}\mathbf{Alg} \qquad (X,\xi) \models A, C \vdash t \equiv t'$$

$$\Updownarrow$$

$$(T_{\mathbb{S}}C, \sigma_C) \models A, C \vdash t \equiv t'$$

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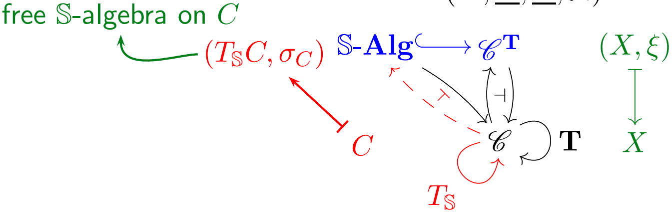

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$$\Updownarrow$$

$$q_C \circ t = q_C \circ t' : \; A \underset{t'}{\overset{t}{\rightrightarrows}} TC \xrightarrow{q_C} T_{\mathbb{S}}C$$

where the quotient map q_C is given by

$$\begin{array}{ccc} TTC & \xrightarrow{Tq_C} & TT_{\mathbb{S}}C \\ \mu_C \downarrow & & \downarrow \sigma_C \\ TC & \xrightarrow{\exists!\, q_C} & T_{\mathbb{S}}C \\ \eta_C \uparrow & \nearrow_{\eta_{\mathbb{S}C}} & \\ C & & \end{array}$$

Theory: Construction of free models

For a TES $\mathbb{S} = (\mathscr{V}, \underline{\mathscr{C}}, \underline{\mathbf{T}}, \mathcal{A})$

The theory of Equational System (Fiore & Hur, ICALP 07) provides an explicit construction of free $\mathbb{S}$-algebras and the quotient map $q : T \dot{\to} T_{\mathbb{S}}$.

Theory: Construction of free models

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The theory of Equational System (Fiore & Hur, ICALP 07) provides an explicit construction of free $\mathbb{S}$-algebras and the quotient map $q : T \dot{\to} T_{\mathbb{S}}$.

If

- $\mathscr{C}$ is cocomplete.
- T preserves colimits of ω-chains and epimorphisms.
- For every $(B, D \vdash s \equiv s') \in \mathcal{A}$, $\underline{\mathscr{C}}(D, -) \otimes B$ preserves colimits of ω-chains and epimorphisms.

For a TES $\mathbb{S} = (\mathscr{V}, \underline{\mathscr{C}}, \underline{\mathbf{T}}, \mathcal{A})$

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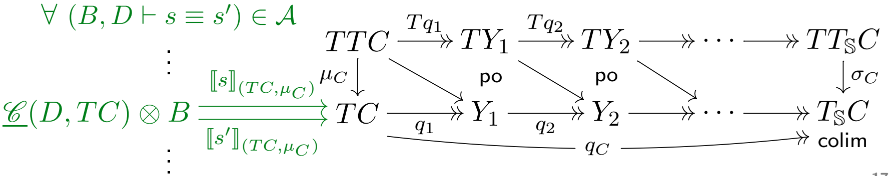
Then $(T_{\mathbb{S}}C, \sigma_C)$ and $q_C : TC \to T_{\mathbb{S}}C$ can be constructed as follows:



Theory: Construction of free models

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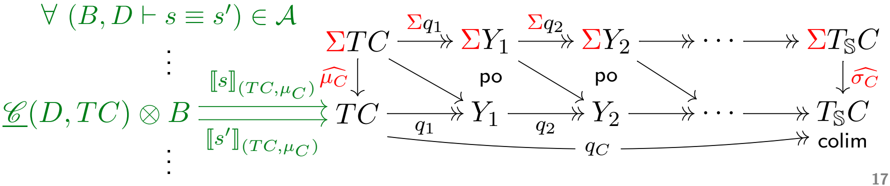
The theory of Equational System (Fiore & Hur, ICALP 07) provides an explicit construction of free $\mathbb{S}$-algebras and the quotient map $q : T \dot{\to} T_{\mathbb{S}}$.

If

- $\mathscr{C}$ is cocomplete.
- Σ preserves colimits of ω-chains and epimorphisms.
- For every $(B, D \vdash s \equiv s') \in \mathcal{A}$, $\underline{\mathscr{C}}(D, -) \otimes B$ preserves colimits of ω-chains and epimorphisms.
- $\mathbf{T}$ is a free monad of an endofunctor Σ.

Then $(T_{\mathbb{S}}C, \sigma_C)$ and $q_C : TC \to T_{\mathbb{S}}C$ can be constructed as follows:



Theory: Towards Completeness

Internal Completeness

$$\mathbb{S}\text{-}\mathbf{Alg} \ \models \ A, C \vdash t \equiv t'$$

$$\Updownarrow$$

$$q_C \circ t = q_C \circ t' : \ A \underset{t'}{\overset{t}{\rightrightarrows}} TC \xrightarrow{q_C} T_{\mathbb{S}}C$$

$+$

Explicit construction of $T_{\mathbb{S}}C$ and q_C

$\Downarrow$

We may synthesise a sound and complete logic.

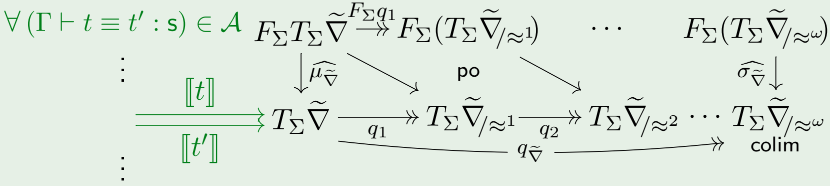
Theory: Towards Completeness

Example (continued)

$$\mathbb{S}\text{-}\mathbf{Alg} \ \models \ \nabla \vdash u \equiv v : \mathsf{r} \quad \overset{\text{Int Comp}}{\Longleftrightarrow} \quad \widetilde{\mathsf{r}} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma \widetilde{\nabla} \xrightarrow{q_{\widetilde{\nabla}}} T_{\Sigma\mathbb{S}} \widetilde{\nabla}$$

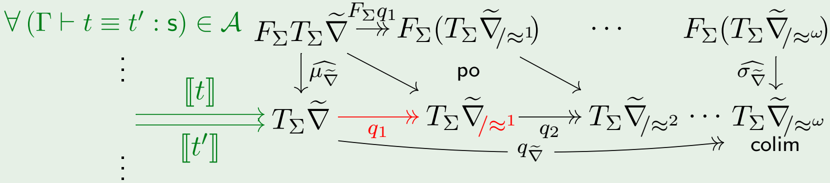
Example (continued)

$\mathbb{S}$-**Alg** $\models \nabla \vdash u \equiv v : \mathsf{r} \quad \overset{\text{Int Comp}}{\Longleftrightarrow} \quad \widetilde{\mathsf{r}} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma \widetilde{\nabla} \xrightarrow{q_{\widetilde{\nabla}}} T_{\Sigma\mathbb{S}} \widetilde{\nabla}$



Example (continued)

$\mathbb{S}\text{-}\mathbf{Alg} \models \nabla \vdash u \equiv v : \mathsf{r} \quad \overset{\text{Int Comp}}{\Longleftrightarrow} \quad \widetilde{\mathsf{r}} \overset{u}{\underset{v}{\rightrightarrows}} T_\Sigma \widetilde{\nabla} \xrightarrow{q_{\widetilde{\nabla}}} T_{\Sigma_{\mathbb{S}}} \widetilde{\nabla}$



$\approx^1_{\mathsf{s}}$ on $T_\Sigma \widetilde{\nabla}(\mathsf{s}) = \{\, t \mid \nabla \vdash t : \mathsf{s} \,\}$

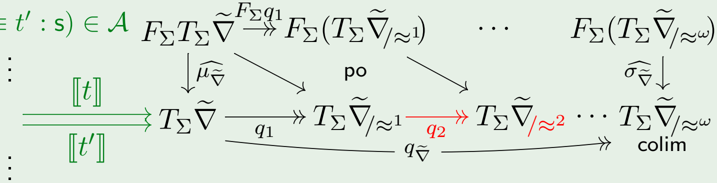
$$\text{Ref}^1 \ \frac{}{t \approx^1_{\mathsf{s}} t} \qquad \text{Sym}^1 \ \frac{t \approx^1_{\mathsf{s}} t'}{t' \approx^1_{\mathsf{s}} t} \qquad \text{Trans}^1 \ \frac{t \approx^1_{\mathsf{s}} t' \quad t' \approx^1_{\mathsf{s}} t''}{t' \approx^1_{\mathsf{s}} t}$$

$$\text{Axiom}^1 \ \frac{(\Gamma \vdash t \equiv t' : \mathsf{s}) \in \mathcal{A} \qquad \{\, \nabla \vdash s_i : \mathsf{s}_i \,\}_{x_i : \mathsf{s}_i \in \Gamma}}{t[x_i \mapsto s_i] \approx^1_{\mathsf{s}} t'[x_i \mapsto s_i]}$$

Example (continued)

$\mathbb{S}\text{-}\mathbf{Alg} \models \nabla \vdash u \equiv v : \mathsf{r} \overset{\text{Int Comp}}{\iff} \widetilde{\mathsf{r}} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma \widetilde{\nabla} \xrightarrow{q_{\widetilde{\nabla}}} T_{\Sigma\mathbb{S}} \widetilde{\nabla}$

$\forall\, (\Gamma \vdash t \equiv t' : \mathsf{s}) \in \mathcal{A}$



$\approx^2_{\mathsf{s}}$ on $T_\Sigma \widetilde{\nabla}(\mathsf{s}) = \{\, t \mid \nabla \vdash t : \mathsf{s} \,\}$

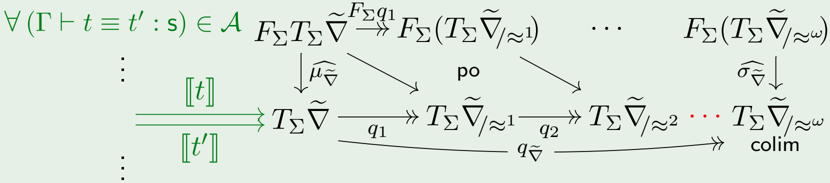
$$\mathsf{Ref}^2 \ \frac{}{t \approx^2_{\mathsf{s}} t} \qquad \mathsf{Sym}^2 \ \frac{t \approx^2_{\mathsf{s}} t'}{t' \approx^2_{\mathsf{s}} t} \qquad \mathsf{Trans}^2 \ \frac{t \approx^2_{\mathsf{s}} t' \quad t' \approx^2_{\mathsf{s}} t''}{t' \approx^2_{\mathsf{s}} t}$$

$$\mathsf{Cong}^2 \ \frac{t_i \approx^1_{\mathsf{s}_i} t'_i \quad (1 \le i \le k)}{f(t_1 .. t_k) \approx^2_{\mathsf{s}} f(t'_1 .. t'_k)} \ f : \mathsf{s}_1 .. \mathsf{s}_k \to \mathsf{s} \in \Sigma \qquad \mathsf{Ext}^2 \ \frac{t \approx^1_{\mathsf{s}} t'}{t \approx^2_{\mathsf{s}} t'}$$

Theory: Towards Completeness

Example (continued)

$$\mathbb{S}\text{-}\mathbf{Alg} \models \nabla \vdash u \equiv v : \mathsf{r} \quad \overset{\text{Int Comp}}{\Longleftrightarrow} \quad \widetilde{\mathsf{r}} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma \widetilde{\nabla} \xrightarrow{q_{\widetilde{\nabla}}} T_{\Sigma\mathbb{S}} \widetilde{\nabla}$$



$\approx^n_{\mathsf{s}}$ on $T_\Sigma \widetilde{\nabla}(\mathsf{s}) = \{\, t \mid \nabla \vdash t : \mathsf{s} \,\}$

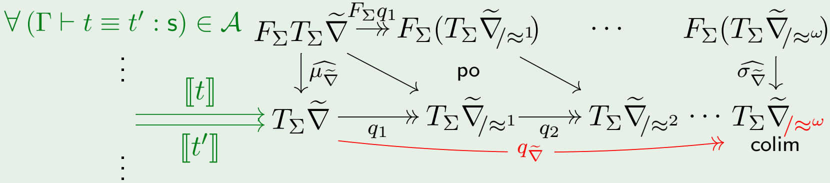
$$\text{Ref}^n \frac{}{t \approx^n_{\mathsf{s}} t} \qquad \text{Sym}^n \frac{t \approx^n_{\mathsf{s}} t'}{t' \approx^n_{\mathsf{s}} t} \qquad \text{Trans}^n \frac{t \approx^n_{\mathsf{s}} t' \quad t' \approx^n_{\mathsf{s}} t''}{t' \approx^n_{\mathsf{s}} t}$$

$$\text{Cong}^n \frac{t_i \approx^{n-1}_{\mathsf{s}_i} t'_i \quad (1 \le i \le k)}{f(t_1..t_k) \approx^n_{\mathsf{s}} f(t'_1..t'_k)}\; f : \mathsf{s}_1..\mathsf{s}_k \to \mathsf{s} \in \Sigma \qquad \text{Ext}^n \frac{t \approx^{n-1}_{\mathsf{s}} t'}{t \approx^n_{\mathsf{s}} t'}$$

Theory: Towards Completeness

Example (continued)

$\mathbb{S}\text{-}\mathbf{Alg} \models \nabla \vdash u \equiv v : \mathsf{r}$ $\overset{\text{Int Comp}}{\iff}$ $\widetilde{\mathsf{r}} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma \widetilde{\nabla} \xrightarrow{q_{\widetilde{\nabla}}} T_{\Sigma\mathbb{S}} \widetilde{\nabla}$



$\approx^\omega_{\mathsf{s}}$ on $T_\Sigma \widetilde{\nabla}(\mathsf{s}) = \{\, t \mid \nabla \vdash t : \mathsf{s} \,\}$

$$\text{Ref}^\omega\ \frac{}{t \approx^\omega_{\mathsf{s}} t} \qquad \text{Sym}^\omega\ \frac{t \approx^\omega_{\mathsf{s}} t'}{t' \approx^\omega_{\mathsf{s}} t} \qquad \text{Trans}^\omega\ \frac{t \approx^\omega_{\mathsf{s}} t' \quad t' \approx^\omega_{\mathsf{s}} t''}{t' \approx^\omega_{\mathsf{s}} t}$$

$$\text{Axiom}^\omega\ \frac{(\Gamma \vdash t \equiv t' : \mathsf{s}) \in \mathcal{A} \qquad \{\, \nabla \vdash s_i : \mathsf{s}_i \,\}_{x_i : \mathsf{s}_i \in \Gamma}}{t[x_i \mapsto s_i] \approx^\omega_{\mathsf{s}} t'[x_i \mapsto s_i]}$$

$$\text{Cong}^\omega\ \frac{t_i \approx^\omega_{\mathsf{s}_i} t'_i \quad (1 \le i \le k)}{f(t_1 .. t_k) \approx^\omega_{\mathsf{s}} f(t'_1 .. t'_k)}\ f : \mathsf{s}_1 .. \mathsf{s}_k \to \mathsf{s} \in \Sigma$$

Theory: Towards Completeness

$$\mathbb{S}\text{-}\mathbf{Alg} \models \nabla \vdash u \equiv v : \mathsf{r}$$

$$\Updownarrow$$

$$\widetilde{\mathsf{r}} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma \widetilde{\nabla} \xrightarrow{q_{\widetilde{\nabla}}} T_\Sigma \widetilde{\nabla}_{/\approx^\omega}$$

$$\Updownarrow$$

$$u \approx^\omega_{\mathsf{r}} v$$

Term Rewriting $\Updownarrow$

$$u \overset{\overset{1}{\sim}_{\mathsf{r}}}{\longleftrightarrow} t_1 \overset{\overset{1}{\sim}_{\mathsf{r}}}{\longleftrightarrow} \cdots \overset{\overset{1}{\sim}_{\mathsf{r}}}{\longleftrightarrow} t_{n-1} \overset{\overset{1}{\sim}_{\mathsf{r}}}{\longleftrightarrow} v$$

$$\text{Ref}^\omega \frac{}{t \approx^\omega_{\mathsf{s}} t} \qquad \text{Sym}^\omega \frac{t \approx^\omega_{\mathsf{s}} t'}{t' \approx^\omega_{\mathsf{s}} t} \qquad \text{Trans}^\omega \frac{t \approx^\omega_{\mathsf{s}} t' \quad t' \approx^\omega_{\mathsf{s}} t''}{t' \approx^\omega_{\mathsf{s}} t}$$

$$\text{Axiom}^\omega \frac{(\Gamma \vdash t \equiv t' : \mathsf{s}) \in \mathcal{A} \qquad \{ \nabla \vdash s_i : \mathsf{s}_i \}_{x_i : \mathsf{s}_i \in \Gamma}}{t[x_i \mapsto s_i] \approx^\omega_{\mathsf{s}} t'[x_i \mapsto s_i]}$$

$$\text{Cong}^\omega \frac{t_i \approx^\omega_{\mathsf{s}_i} t'_i \quad (1 \le i \le k)}{f(t_1 .. t_k) \approx^\omega_{\mathsf{s}} f(t'_1 .. t'_k)} \; f : \mathsf{s}_1 .. \mathsf{s}_k \to \mathsf{s} \in \Sigma$$

$$\mathbb{S}\text{-}\mathbf{Alg} \models \nabla \vdash u \equiv v : \mathsf{r}$$

$$\Uparrow$$

Completeness

As proofs of $u \approx^{\omega}_{\mathsf{r}} v$ can be turned into proofs in MFoEL,

MFoEL is complete.

$$u \overset{\approx_{\mathsf{r}}}{\longleftrightarrow} t_1 \overset{\approx_{\mathsf{r}}}{\longleftrightarrow} \cdots \overset{\approx_{\mathsf{r}}}{\longleftrightarrow} t_{n-1} \overset{\approx_{\mathsf{r}}}{\longleftrightarrow} v$$

$$\text{Ref}^{\omega}\ \frac{}{t \approx^{\omega}_{\mathsf{s}} t} \qquad \text{Sym}^{\omega}\ \frac{t \approx^{\omega}_{\mathsf{s}} t'}{t' \approx^{\omega}_{\mathsf{s}} t} \qquad \text{Trans}^{\omega}\ \frac{t \approx^{\omega}_{\mathsf{s}} t' \quad t' \approx^{\omega}_{\mathsf{s}} t''}{t' \approx^{\omega}_{\mathsf{s}} t}$$

$$\text{Axiom}^{\omega}\ \frac{(\Gamma \vdash t \equiv t' : \mathsf{s}) \in \mathcal{A} \qquad \{\, \nabla \vdash s_i : \mathsf{s}_i \,\}_{x_i : \mathsf{s}_i \in \Gamma}}{t[x_i \mapsto s_i] \approx^{\omega}_{\mathsf{s}} t'[x_i \mapsto s_i]}$$

$$\text{Cong}^{\omega}\ \frac{t_i \approx^{\omega}_{\mathsf{s}_i} t'_i \quad (1 \leq i \leq k)}{f(t_1..t_k) \approx^{\omega}_{\mathsf{s}} f(t'_1..t'_k)}\ f : \mathsf{s}_1..\mathsf{s}_k \to \mathsf{s} \in \Sigma$$

Subtlety of MFoEL [Goguen & Meseguer 1985]

Example (continued)

$$
\begin{aligned}
\mathsf{S} &= \{ \mathbf{F}, \mathbf{B} \} \\
\Sigma &= \{ \mathsf{True} : \mathbf{B}, \quad \mathsf{False} : \mathbf{B}, \quad \neg : \mathbf{B} \to \mathbf{B}, \\
&\qquad \wedge : \mathbf{B}, \mathbf{B} \to \mathbf{B}, \quad \vee : \mathbf{B}, \mathbf{B} \to \mathbf{B}, \\
&\qquad \mathsf{Foo} : \mathbf{F} \to \mathbf{B} \} \\
\mathcal{A} &= \{ \vdash \neg\mathsf{True} = \mathsf{False} : \mathbf{B}, \quad \vdash \neg\mathsf{False} = \mathsf{True} : \mathbf{B}, \\
&\qquad x : \mathbf{B} \vdash x \vee \neg x = \mathsf{True} : \mathbf{B}, \quad x : \mathbf{B} \vdash x \wedge \neg x = \mathsf{False} : \mathbf{B}, \\
&\qquad x : \mathbf{B} \vdash x \vee x = x : \mathbf{B}, \quad x : \mathbf{B} \vdash x \wedge x = x : \mathbf{B}, \\
&\qquad y : \mathbf{F} \vdash \mathsf{Foo}(y) = \neg\mathsf{Foo}(y) : \mathbf{B} \}
\end{aligned}
$$

Subtlety of MFoEL [Goguen & Meseguer 1985]

Example (continued)

$$
\begin{aligned}
\mathsf{S} &= \{\, \mathbf{F}, \mathbf{B} \,\} \\
\Sigma &= \{\ \mathsf{True} : \mathbf{B},\quad \mathsf{False} : \mathbf{B},\quad \neg : \mathbf{B}{\to}\mathbf{B}, \\
&\qquad \wedge : \mathbf{B}, \mathbf{B}{\to}\mathbf{B},\quad \vee : \mathbf{B}, \mathbf{B}{\to}\mathbf{B}, \\
&\qquad \mathsf{Foo} : \mathbf{F}{\to}\mathbf{B} \ \} \\
\mathcal{A} &= \{ \vdash \neg\mathsf{True} = \mathsf{False} : \mathbf{B},\quad \vdash \neg\mathsf{False} = \mathsf{True} : \mathbf{B}, \\
&\qquad x : \mathbf{B} \vdash x \vee \neg x = \mathsf{True} : \mathbf{B},\quad x : \mathbf{B} \vdash x \wedge \neg x = \mathsf{False} : \mathbf{B}, \\
&\qquad x : \mathbf{B} \vdash x \vee x = x : \mathbf{B},\quad x : \mathbf{B} \vdash x \wedge x = x : \mathbf{B}, \\
&\qquad y : \mathbf{F} \vdash \mathsf{Foo}(y) = \neg\mathsf{Foo}(y) : \mathbf{B} \ \}
\end{aligned}
$$

False reasoning

Theorem: $\vdash \mathsf{True} = \mathsf{False} : \mathbf{B}$

Proof:

$$
\begin{aligned}
\mathsf{True} &= \mathsf{Foo}(y) \vee \neg\mathsf{Foo}(y) = \mathsf{Foo}(y) \vee \mathsf{Foo}(y) = \mathsf{Foo}(y) \\
&= \mathsf{Foo}(y) \wedge \mathsf{Foo}(y) = \mathsf{Foo}(y) \wedge \neg\mathsf{Foo}(y) = \mathsf{False}
\end{aligned}
$$

Example (continued)

$\mathsf{S} = \{\,\mathbf{F}, \mathbf{B}\,\}$

$\Sigma = \{\ \mathsf{True} : \mathbf{B},\quad \mathsf{False} : \mathbf{B},\quad \neg : \mathbf{B}{\to}\mathbf{B},$

$\wedge : \mathbf{B}, \mathbf{B}{\to}\mathbf{B},\quad \vee : \mathbf{B}, \mathbf{B}{\to}\mathbf{B},$

F

$\mathcal{A} = \{\,\vdash$

x ... se $: \mathbf{B}$,

x

y

For a judgement $\nabla \vdash u \equiv v : \mathsf{r}$

$\approx^{\omega}_{\mathsf{r}}$ is a relation on the set $\{\, t \mid \nabla \vdash t : \mathsf{r}\,\}$

False reasoning

Theorem: $\vdash \mathsf{True} = \mathsf{False} : \mathbf{B}$

Proof:

$$\begin{aligned}\mathsf{True} &= \mathsf{Foo}(y) \vee \neg\mathsf{Foo}(y) = \mathsf{Foo}(y) \vee \mathsf{Foo}(y) = \mathsf{Foo}(y)\\ &= \mathsf{Foo}(y) \wedge \mathsf{Foo}(y) = \mathsf{Foo}(y) \wedge \neg\mathsf{Foo}(y) = \mathsf{False}\end{aligned}$$

Example (continued)

$\mathsf{S} = \{ \mathbf{F}, \mathbf{B} \}$

$\Sigma = \{ \mathsf{True} : \mathbf{B},\quad \mathsf{False} : \mathbf{B},\quad \neg : \mathbf{B} {\rightarrow} \mathbf{B},$

$\quad \wedge : \mathbf{B}, \mathbf{B} {\rightarrow} \mathbf{B},\quad \vee : \mathbf{B}, \mathbf{B} {\rightarrow} \mathbf{B},$

$\quad \mathsf{F}$ …

$\mathcal{A} = \{ \vdash$ … $\mathsf{se} : \mathbf{B},$

$\quad x$ …

$\quad x$ …

$\quad y : \mathbf{F} \vdash \mathsf{Foo}(y) = \neg\mathsf{Foo}(y) : \mathbf{B} \}$

For a judgement $\nabla \vdash u \equiv v : \mathsf{r}$

$\approx^{\omega}_{\mathsf{r}}$ is a relation on the set $\{ t \mid \nabla \vdash t : \mathsf{r} \}$

Valid reasoning

Theorem: $y : \mathbf{F} \vdash \mathsf{True} = \mathsf{False} : \mathbf{B}$

Proof:

$$\mathsf{True} = \mathsf{Foo}(y) \vee \neg\mathsf{Foo}(y) = \mathsf{Foo}(y) \vee \mathsf{Foo}(y) = \mathsf{Foo}(y)$$

$$= \mathsf{Foo}(y) \wedge \mathsf{Foo}(y) = \mathsf{Foo}(y) \wedge \neg\mathsf{Foo}(y) = \mathsf{False}$$

Concluding Remarks

Applications

- Multi-sorted first-order equational logic $(\boldsymbol{\mathcal{S}et}, \boldsymbol{\mathcal{S}et}^{\mathsf{S}})$
- Nominal equational logic $(\boldsymbol{\mathcal{N}om}, \underline{\boldsymbol{\mathcal{N}om}})$
  (Gabbay & Mathijssen 06; Clouston & Pitts 07)
- Binding equational logic $(\boldsymbol{\mathcal{S}et}^{\mathbb{I}}, \underline{\boldsymbol{\mathcal{S}et}^{\mathbb{I}}})$
  (Hamana 03)
- Second-order equational logic $(\boldsymbol{\mathcal{S}et}^{\mathbb{F}}, \underline{\boldsymbol{\mathcal{S}et}^{\mathbb{F}}})$
  in the context of second-order abstract syntax (Fiore 08)

Extension

- Term Equational Rewriting Systems and Logics
  e.g. Rewriting logic (Meseguer et al.)