Implications of Yoneda Lemma to Category Theory

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Abstract

This is a survey paper on the implication of Yoneda lemma, named after Japanese mathematician Nobuo Yoneda, to category theory. We prove Yoneda lemma. We use Yoneda lemma to prove that each of the notions universal morphism, universal element, and representable functor subsumes the other two. We prove that a category is anti-equivalent to the category of its representable functors as a corollary of Yoneda lemma. We also prove the Yoneda embedding, i.e. representable functors are isomorphic if and only if their representers are isomorphic.

1 Introduction

This is a survey paper on the implication of Yoneda lemma to category theory. We prove Yoneda lemma. Yoneda lemma is a pure abstract non-sense based solely on categorical arguments. However, it has very non-trivial implication to category theory, and in turn, whole mathematics.

Universals and universal properties are one of the most useful tools in homological algebra. For example, free algebraic structures, products and coproducts, direct and inverse limits, tensor products, (categorical) kernels, images, monomorphisms, and epimorphisms are defined by their universal properties. Universal elements and universal morphisms are typical constructions of universals.

Representable functors are those functors isomorphic to a morphism functor. Since morphism functors arises naturally in many situations, representable functors enjoy nice properties. We use Yoneda lemma to prove that each of the notions universal morphism, universal element, and representable functor subsumes the other two.

As an another corollary of Yoneda lemma, we prove that a category is anti-equivalent to the category of its representable functors. We also prove that representable functors are isomorphic if and only if their representers are isomorphic. Thus this embedding of an object into its representation is called the Yoneda embedding.

The material is largely based on [1].

2 Categories, Functors, and Natural Transformations

Definition 1 (Category)**.** A *category* C consists of a set A of objects, set $\mathrm{Mor}_C(a,b)$ of morphisms from a to b for each $a,b \in A$, and composition $\circ : \mathrm{Mor}_C(b,c) \times \mathrm{Mor}_C(a,b) \to \mathrm{Mor}_C(a,c)$ for each $a,b,c \in A$ such that

1. (associativity) for all $f \in \mathrm{Mor}_C(a,b)$, $g \in \mathrm{Mor}_C(b,c)$, and $h \in \mathrm{Mor}_C(c,d)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$ and

2. (identity) for all object $a \in A$, there exists a morphism $1_a \in \mathrm{Mor}_C(a, a)$ such that for all $f \in \mathrm{Mor}_C(b, c)$, $1_c \circ f = f$ and $f \circ 1_b = f$.

Example 2 (One-Object Category)**.** Let $*$ be the one-object category consisting of

1. set $\{*\}$ of objects and

2. set $\mathrm{Mor}_*(*, *) = \{1_*\}$ of morphisms.

It is trivially a category. We abuse $*$ to denote the one-element set, the one-object category, and its only object. It is clear from the context to distinguish.

Notation 3. Note that we abuse $\circ$ to denote all compositions. By $f : a \to b$ is meant $f \in \mathrm{Mor}_C(a, b)$ for each $a, b \in A$, if C is unambiguous from the context. For a category C, we abuse $c \in C$ to denote c is an object of C.

Lemma 4. *Identity is unique for each object.*

Proof. Let C be a category. For all $a \in C$, let $1_a, 1'_a$ be two identities. We have

$$1_a = 1_a \circ 1'_a = 1'_a.$$

$\square$

Definition 5 (Isomorphism)**.** Let C be a category and c, c' be objects. Objects c and c' are *isomorphic* if there exists $f : c \to c'$ and $g : c' \to c$ such that $g \circ f = 1_c$ and $f \circ g = 1_{c'}$. In such a case, f and g are called *isomorphism*.

Definition 6 (Initial Object)**.** For a category C, an object $c \in C$ is *initial* if for all $c' \in C$, there exists a unique morphism $f : c \to c'$.

Lemma 7 (Unique Initial Object)**.** *For a category C, initial objects are unique up to isomorphism.*

Proof. Let $c, c' \in C$ be initial objects. By Definition 6, there exists unique $f : c \to c'$ and $g : c' \to c$. By the uniqueness of morphisms, $g \circ f : c \to c$ should be 1_c and $f \circ g = 1_{c'}$. Thus c and c' are isomorphic. $\square$

Definition 8 (Opposite Category)**.** For a category C, let C^{op} be the opposite category of C consisting of

1. set of objects in the category C as the set of objects and

2. $\mathrm{Mor}_{C^{\mathrm{op}}}(a, b) = \mathrm{Mor}_C(b, a)$

with composition $g \circ_{C^{\mathrm{op}}} f = f \circ_C g$. It is trivially a category.

Definition 9 (Functor)**.** For categories C and B, a *functor* $T : C \to B$ consists of

1. the object function T which assigns to each object $c \in C$ an object $Tc \in D$ and

2. the morphism function T which assigns to each morphism $f : a \to b$ a morphism $Tf : Ta \to Tb$

such that

$$T(1_c) = 1_{Tc} \text{ and } T(g \circ f) = Tg \circ Tf.$$

Example 10 (Functor from One-Object Category)**.** A functor from the one-object category $*$ to C is effectively an element in C. To see this, correspond a functor $T : * \to C$ to an element $T* \in C$.

Definition 11 (Full Functor, Faithful Functor)**.** Let C, B be categories and $T : C \to B$ be a functor. Functor T is *full* if $T : \mathrm{Mor}_C(a, b) \to \mathrm{Mor}_B(Ta, Tb)$ is surjective for all $a, b \in C$. Functor T is *faithful* if $T : \mathrm{Mor}_C(a, b) \to \mathrm{Mor}_B(Ta, Tb)$ is injective for all $a, b \in C$.

Definition 12 (Natural Transformation)**.** For categories C and B and functors $S, T : C \to B$, *natural transformation* $\tau : S \dot{\to} T$ consists of $\tau_c : Sc \to Tc$ for each c such that the following diagram commutes:

$$\begin{tikzcd} c \arrow[d, "f"] & Sc \arrow[r, "\tau_c"] \arrow[d, "Sf"] & Tc \arrow[d, "Tf"] \\ c' & Sc' \arrow[r, "\tau_{c'}"] & Tc' \end{tikzcd}$$

Definition 13 (Natural Isomorphism)**.** For categories C and B and functors $S, T : C \to B$, *natural isomorphism* $\tau : S \dot{\to} T$ is a natural transformation consisting of $\tau_c : Sc \to Tc$ for each c such that τ_c is an isomorphism for each c.

Definition 14 (Equivalent Category)**.** Categories C, B are *equivalent* if there exists functors $T : C \to B$ and $S : B \to C$ such that $S \circ T$ is naturally isomorphic to 1_C and $T \circ S$ is naturally isomorphic to 1_B.

Definition 15 (Anti-Equivalent Category)**.** Categories C, B are *anti-equivalent* if C and B^{op} are equivalent.

Eilenberg-Mac Lane observed that "*category* has been defined to be able to define *functor* and *fuctor* has been defined in order to be able to define *natural transformation*" [1].

3 Foundations

To define the categories of sets, categories, and functors, we have to address set-theoretical constructions and mathematical foundations. The famous Russell paradox implies that we have to confine the comprehension principle: forming the set $\{x|\phi(x)\}$ from a given property $\phi(x)$. The category of categories meets the same problem: how to define it from the axiomatic property of categories?

We avoid this problem by introducing the *universe* U. By considering only sets in the universe and confining the comprehension principle to the universe, we detour the Russell paradox.

Axiom 16 (Set Theory)**.** We assume the following set-theoretical axioms:

1. For sets u and v, the *unordered pair* $\{u, v\}$ is also a set.
2. There exists the *infinite set* $\omega = \{0, 1, 2, \ldots\}$ *of all finite ordinals.*
3. For a set u and v, the *Cartesian product* $u \times v = \{(x, y)|x \in u, y \in v\}$ is also a set. Note that the ordered pair $(x, y) = \{x, \{x, y\}\}$.
4. For a set u, the *power set* $\wp u = \{v|v \subseteq u\}$ is also a set.
5. For a set u, the *union* $\cup u = \{y|y \in x \text{ for some } x \in u\}$ is also a set.

6. For a set u and a property $\phi(x)$, the *comprehension* $\{x|x \in u, \phi(x)\}$ is also a set.
7. There exists the *universe* U such that
   a) the universe U is closed under the construction of the unordered pair, the infinite set of all finite ordinals, the Cartesian product, the power set, and the union,
   b) if $x \in u \in U$ then $x \in U$, and
   c) if $f : a \to b$ is surjective, $a \in U$, and $b \subseteq U$ then $b \in U$.

By a *small set* is meant an element of the universe U.

4 Categories of (Small) Sets, Categories, and Functors

Thanks to appropriate mathematical foundations, we are able to define categories of (small) sets, (small) categories, and (small) functors.

Lemma 17. *Small sets form the category **Set** with functions as morphisms.*

Proof. Let a and b be small sets and $f : a \to b$ be a function. Note that the set-theoretical representation of f is small thanks to the axioms on the universe U, hence the name 'category of small sets' is legitimate. Ordinary function composition serves as the legitimate composition for set morphisms. Hence **Set** is indeed a category. □

Definition 18 (Small Category)**.** A *small category* is a category C with small set A of objects and small set $\mathrm{Mor}_C(a, b)$ of morphisms for each $a, b \in A$.

Lemma 19. *Small categories form the category **Cat** with functors as morphisms.*

Proof. Let C and B be small categories and $T : C \to B$ be a functor. Note that the set-theoretical representation of T is small thanks to the axioms on the universe U, hence the name 'category of small categories' is legitimate.

Let D, C, and B be small categories and let $T : D \to C$ and $S : C \to B$ be functors. The composition $S \circ T : D \to B$ with

1. object function $S \circ T : d \mapsto S(Td)$ and
2. morphism function $S \circ T : f \mapsto S(Tf)$

is a functor as follows:

1. $(ST)(1_d) = S(T(1_d)) = S(1_{Td}) = 1_{S(Td)} = 1_{(ST)(d)}$ and
2. $(ST)(g \circ f) = S(T(g \circ f)) = S(Tg \circ Tf) = S(Tg) \circ S(Tf) = (ST)(g) \circ (ST)(f)$.

For each small category C, let 1_C be the identity functor with

1. object function $1_C : c \mapsto c$ and
2. morphism function $1_C : f \mapsto f$.

It is indeed a functor as follows:

1. $1_C(1_c) = 1_c = 1_{1_C(c)}$ and
2. $1_C(g \circ f) = g \circ f = 1_C(g) \circ 1_C(f)$.

Let C and B be small categories and $T : C \to B$ be a functor. We have

$$1_B \circ T = T = T \circ 1_C.$$

Let E, D, C, and B be small categories and $R : C \to B$, $S : D \to C$, and $T : E \to D$ be functors. We have

$$(R \circ S) \circ T = R \circ (S \circ T).$$

Hence **Cat** is indeed a category. □

Lemma 20. *Let C and B be small categories. Functors from C to B form the category B^C with natural transformations as morphisms.*

Proof. Let $R, S, T : C \to B$ be functors and let $\alpha : R \dot{\to} S, \beta : S \dot{\to} T$ be natural transformations. The composition $\beta \circ \alpha$ with $(\beta \circ \alpha)_c = \beta_c \circ \alpha_c$ is a natural transformation from R to T as follows:

$$\begin{tikzcd} c \arrow[d, "f"] & Rc \arrow[r, "\alpha_c"] \arrow[d, "Rf"] & Sc \arrow[r, "\beta_c"] \arrow[d, "Sf"] & Tc \arrow[d, "Tf"] \\ c' & Rc' \arrow[r, "\alpha_{c'}"] & Sc' \arrow[r, "\beta_{c'}"] & Tc' \end{tikzcd}$$

For each functor $T : C \to B$, let $1_T : T \dot{\to} T$ be the identity natural transformation with $(1_T)_c = 1_{(Tc)}$. It is indeed a natural transformation as follows:

$$\begin{tikzcd} c \arrow[d, "f"] & Tc \arrow[r, "(1_T)_c"] \arrow[d, "Tf"] & Tc \arrow[d, "Tf"] \\ c' & Tc' \arrow[r, "(1_T)_{c'}"] & Tc' \end{tikzcd}$$

Let $S, T : C \to B$ be functors and $\alpha : S \dot{\to} T$ be a natural transformation. We have

$$1_T \circ \alpha = \alpha = \alpha \circ 1_S.$$

Let $Q, R, S, T : C \to B$ be functors and $\alpha : S \dot{\to} T$, $\beta : R \dot{\to} S$, $\gamma : Q \dot{\to} R$ be natural transformations. We have

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma).$$

Hence B^C is indeed a category. □

Note that the set of morphisms of a functor category is not necessarily small.

5 Yoneda Lemma

First, we present the (covariant) morphism functor. It enables us to state Yoneda Lemma and prove reducibility of the notion *universal arrow* to the notion *universal object*.

Lemma 21 (Morphism Functor)**.** *Let C be a small category with small morphism sets, i.e., for all $a, b \in C$, $Mor_C(a, b)$ is a small set. Let a be an object in C. The following two functions form the (covariant) morphism functor $C(a, -) : C \to$ **Set**.*

1. *object function $C(a, -) : b \mapsto Mor_C(a, b)$ and*

2. *morphism function* $C(a,-) : f \mapsto (C(a,f) : g \mapsto f \circ g)$

Proof. 1. $C(a,1_b)(g) = 1_b \circ g = g$, so $C(a,1_b) = 1_{\mathrm{Mor}_C(a,b)} = 1_{C(a,b)}$ and

2. $(C(a,g) \circ C(a,f))(h) = C(a,g)(f \circ h) = g \circ f \circ h = C(a, g \circ f)(h)$, so $(C(a,g) \circ C(a,f)) = C(a, g \circ f)$.

∎

Notation 22. To be consistent with the notation of the (covariant) morphism functor, by $C(a,b)$ is meant $\mathrm{Mor}_C(a,b)$ for a category C and objects $a, b \in C$.

Lemma 23 (Yoneda Lemma)**.** *Let D be a small category with small morphism sets, r be an object in D, $K : D \to \boldsymbol{Set}$ be a functor, and $y : \boldsymbol{Set}^D(D(r,-), K) \to Kr$ be*

$$y(\alpha) = \alpha_r 1_r$$

for each $\alpha : D(r,-) \dot{\to} K$. Then y is bijective.

Proof. Let $\alpha : D(r,-) \dot{\to} K$. Consider the following commutative diagram for each $d \in D$ and $f : r \to d$:

$$\begin{tikzcd}
r \arrow[d, "f"] & D(r,r) \arrow[r, "\alpha_r"] \arrow[d, "D(r,f)"] & Kr \arrow[d, "Kf"] \\
d & D(r,d) \arrow[r, "\alpha_d"] & Kd.
\end{tikzcd}$$

Especially for $1_r \in D(r,r)$, we have

$$\begin{tikzcd}
r \arrow[d, "f"] & 1_r \arrow[r, "\alpha_r"] \arrow[d, "D(r,f)"] & \alpha_r 1_r \arrow[d, "Kf"] \\
d & f \arrow[r, "\alpha_d"] & \alpha_d f = (Kf)(\alpha_r 1_r).
\end{tikzcd}$$

Let $\alpha, \beta : D(r,-) \dot{\to} K$ be natural transformations such that $y(\alpha) = y(\beta)$, i.e. $\alpha_r 1_r = \beta_r 1_r$. By the commutative diagram above, we have for each $d \in D$ and $f : r \to d$:

$$\alpha_d f = (Kf)(\alpha_r 1_r) = (Kf)(\beta_r 1_r) = \beta_d f.$$

Hence $\alpha = \beta$. Thus y is injective.

Conversely, for any $s \in Kr$, define $\alpha_d : D(r,d) \to Kd$ by

$$\alpha_d : f \mapsto (Kf)(s).$$

We have a commutative diagram for each $d, d' \in D$ and $f : d \to d'$:

$$\begin{tikzcd}
d \arrow[d, "f"] & g \arrow[r, "\alpha_d"] \arrow[d, "D(r,f)"] & (Kg)(s) \arrow[d, "Kf"] \\
d' & f \circ g \arrow[r, "\alpha_{d'}"] & (K(f \circ g))(s) = (Kf)((Kg)(s)).
\end{tikzcd}$$

Hence α is a natural transformation and $\alpha_r 1_r = s$. Thus y is surjective. ∎

6 Universal Morphisms and Universal Objects

In this section, we introduce universal morphisms and universal objects which play a key role in this paper. To introduce universals in an elegant way, we construct *comma categories* based on other categories and functors.

Lemma 24 (Comma Category)**.** *Let C, D, and E be small categories and $T : E \to C$ and $S : D \to C$ be functors. The comma category $T \downarrow S$ consisting of*

1. *set of objects (e, d, f) where $e \in E$, $d \in D$ and $f : Te \to Sd$ and*

$$\begin{tikzcd} Te \arrow[d, "f"] \\ Sd \end{tikzcd}$$

2. *set of morphisms $(k, h) : (e, d, f) \to (e', d', f')$ where the following diagram commutes:*

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"] & Sd' \end{tikzcd}$$

is indeed a small category.

Proof. The set-theoretical representation of $T \downarrow S$ is small thanks to the axioms on the universe U.

Morphisms have a natural composition $(k', h') \circ (k, h) = (k' \circ k, h' \circ h)$ as follows:

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"] & Te' \arrow[r, "Tk'"] \arrow[d, "f'"] & Te'' \arrow[d, "f''"] \\ Sd \arrow[r, "Sh"] & Sd' \arrow[r, "Sh'"] & Sd'' \end{tikzcd}$$

Associativity comes from the associativity of morphism compositions in E and D. We have the identity $1_{(e,d,f)} = (1_e, 1_d)$ for each object (e, d, f). □

Notation 25. Consider functors $T : * \to C$ and $S : D \to C$ and the comma category $T \downarrow S$. Since functor T is effectively an object $T* \in C$, we denote $c \downarrow S$ to mean $T \downarrow S$ where $T : * \to C$ and $T* = c$. Dually for $T \downarrow c$ and eventually for $c \downarrow c'$.

Definition 26 (Universal Morphisms)**.** Let C and D be small categories and $S : D \to C$ be a functor. A *universal morphism* from c to S is an initial object in $c \downarrow S$.

Definition 27 (Universal Objects)**.** Let D be a small category and $S : D \to \mathbf{Set}$ be a functor. A *universal object* of S is a universal morphism from $*$ to S. Here $*$ is the one-element set.

Remark 28. Conversely, "the notion *universal arrow[morphism]* is a special case of the notion *universal element[object]*" [1]. Let C and D be small categories, c an object in C, and $S : D \to C$ be a functor. A universal morphism $(d, r : c \to Sd)$ from c to S is a universal object of the functor $C(c, S-)$.

In many cases, "the function embedding a mathematical object in a suitably completed object can be interpreted as a universal arrow[morphism]," including *free functors* and *fields of quotients* [1].

7 Representable Functors

Definition 29 (Representable Functor)**.** Let D be a small category. A functor $K : D \to \mathbf{Set}$ is *representable* if there exists $d \in D$ and a natural isomorphism $\alpha : D(d, -) \cong K$. Such (d, α) is called a *representation* of K.

Definition 30 (Category of Representable Functors)**.** For a small category C, let R_C be the full subcategory of $\mathbf{Set}^C$ consisting of

1. representable functors as objects and
2. $\mathrm{Mor}_{R_C}(F, G) = \mathrm{Mor}_{\mathbf{Set}^C}(F, G)$ for all representable functors $F, G : C \to \mathbf{Set}$.

It is trivially a category.

Remark 31. Let C and D be small categories and $S : D \to C$ be a functor. A universal morphism from c to S amounts to a natural isomorphism $D(r, -) \cong C(c, S-)$, which in turn is a representation of $C(c, S-)$.

Now universal morphisms can be stated with morphisms.

Lemma 32. *Let C and D be small categories, $c \in C$, and $S : D \to C$ be a functor.*

1. *A pair $(r, u : c \to Sr)$ is a universal morphism from c to S if and only if for each $d \in D$, function $\alpha_d : D(r, d) \to C(c, Sd)$ such that*

$$\alpha_d : f \mapsto Sf \circ u$$

   is bijective. In such a case, α is a natural isomorphism.

2. *Conversely, given r and c, any natural isomorphism $\alpha : D(r, -) \cong C(c, S-)$ is determined in this way by a unique universal morphism (r, d) from c to S.*

Proof. 1. The universal property of (r, u) is exactly for each $d \in D$, α_d being bijective. We have a commutative diagram

$$\begin{tikzcd} d \arrow[d, "g"] & f \arrow[r, "\alpha_d"] \arrow[d, "D(r,g)"] & Sf \circ u \arrow[d, "C(c,Sg)"] \\ d' & g \circ f \arrow[r, "\alpha_{d'}"] & S(g \circ f) \circ u. \end{tikzcd}$$

Hence α is a natural transformation.

2. By Lemma 23, $y : \mathbf{Set}^D(D(r, -), C(c, S-)) \to C(c, Sr)$ such that

$$y(\alpha) = \alpha_r 1_r$$

is bijective. Being a natural isomorphism further requires α_d be bijective for each $d \in D$ in the following diagram:

$$\begin{tikzcd} r \arrow[d, "f"] & 1_r \arrow[r, "\alpha_r"] \arrow[d, "D(r,f)"] & \alpha_r 1_r \arrow[d, "C(c,Sf)"] \\ d & f \arrow[r, "\alpha_d"] & Sf \circ (\alpha_r 1_r). \end{tikzcd}$$

This is exactly $y(\alpha)$ being a universal moprhism from c to S. Hence there exists a bijection from natural isomorphisms from $D(r,-)$ to $C(c,S-)$ and universal morphisms $(r, u : c \to Cr)$.

$\square$

Lemma 33. *Let D be a small category, $K : D \to$ **Set** be a functor, and $(r, u : * \to Kr)$ be a universal arrow from $*$ to K. Let $\alpha_d : D(r,d) \to Kd$ be*

$$\alpha_d : f' \mapsto K(f')(u*).$$

Then α is a representation of K.

Conversely, every representation of K is obtained in this way from exactly one such universal morphism.

Proof. We have $\mathbf{Set}(*,-)$ and $1_{\mathbf{Set}}$ are naturally isomorphic. By composing with K, we have $\mathbf{Set}(*,K-)$ and K are naturally isomorphic. Hence a representation of K amounts to a natural isomorphism between $\mathbf{Set}(*,K-)$ and $D(r,-)$ for some r. Lemma 32 concludes the proof. $\square$

As a corollary of Lemma 23, category k-**CommAlg** of commutative k-algebras for a commutative ring k is anti-equivalent to the category of representable functors for k-**CommAlg** [2].

Lemma 34. *Category D is anti-equivalent to the category R_D of representable functors for the category D.*

Proof. Let $\mathcal{F} : D \to R_D$ be a contravariant functor with

1. object function $\mathcal{F} : r \mapsto D(r,-)$ and
2. morphism function $\mathcal{F} : f \mapsto [u \mapsto u \circ f]$.

It is indeed a contravariant functor as follows:

1. $\mathcal{F}(1_r) = 1_{D(r,r)}$ and
2. $\mathcal{F}(g \circ f) = [u \mapsto u \circ (g \circ f)] = [u \mapsto (u \circ g) \circ f] = \mathcal{F}(f) \circ \mathcal{F}(g)$.

By definition 29, for all $F \in R_D$, there exists $r \in D$ such that $F \cong D(r,-)$. Also, $\mathcal{F}$ is full and faithful. To see this, by Lemma 23, $y : \mathbf{Set}^D(D(r,-), D(s,-)) \to D(s,r)$ defined by

$$y : \alpha \mapsto \alpha_r 1_r$$

is bijective. Thus D and R_D are equivalent by $\mathcal{F}$[1]. $\square$

Lemma 35. *For a category C and objects $c, c' \in C$, $C(c,-)$ and $C(c',-)$ are isomorphic if and only if c and c' are isomorphic.*

[1] In this paper, we have deliberately addressed set-theoretical foundations by introducing a universe. However, in this step, a kind of *the axiom of choice* for as large classes as R_D is required. We do not address this and implicitly appeal to the categorical foundations.

Proof. (⇐) obvious.

(⇒) Let $\alpha : C(c, -) \dot{\to} C(c', -)$ and $\beta : C(c', -) \dot{\to} C(c, -)$ be functors such that $\beta \circ \alpha = 1_{C(c,-)}$ and $\alpha \circ \beta = 1_{C(c',-)}$. By Lemma 23, there exists $f \in C(c, c')$ and $g \in C(c', c)$ such that $\alpha_d : C(c, d) \to C(c', d)$ satisfies

$$\alpha_d : u \mapsto u \circ g$$

and $\beta_d : C(c', d) \to C(c, d)$ satisfies

$$\beta_d : v \mapsto v \circ f.$$

Hence $(\beta \circ \alpha)_c$ maps 1_c to $g \circ f$ and $(\alpha \circ \beta)_{c'}$ maps $1_{c'}$ to $f \circ g$. Since $\beta \circ \alpha$ and $\beta \circ \alpha$ are identity functors, $g \circ f = 1_c$ and $f \circ g = 1_{c'}$. Thus c and c' are isomorphic. □

8 Discussion

Definition 27, Remark 28, Remark 31, and Lemma 33 collectively prove that "each of the notions *universal arrow[morphism]*, *universal element[object]*, and *representable functor* subsumes the other two" [1].

Lemma 34 and Lemma 35 implies that the study of a category and the study of its representable functors are closely related [2]. First of all, the categories are anti-equivalent. Secondly, Yoneda lemma embeds a category into the category of its representable functors, i.e. representable functors are isomorphic if and only if their representers are isomorphic. This is called the *Yoneda embedding* [1].

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