Applications of (In)Equational Systems: Nominal Equational Logic ~~and Nominal Rewriting~~

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Outline of the Talk

1. Definition and Theory of (In)Equational System.
2. Application of Equational System: Nominal Equational Logic
3. ~~Application of Inequational System: Nominal Rewriting~~

Overview: Definition of (In)Equational System

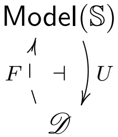
1. **Equational Systems are a framework for systems with equations.**
   - Equational Logics,
   - Computational Calculi such as λ-calculus, π-calculus,
   - Data Types, ...

2. **Inequational Systems are a framework for rewriting systems.**
   - First-order Term Rewriting Systems (TRS),
   - Higher-order Term Rewriting Systems (HRS),
   - Combinatory Reduction Systems (CRS),
   - Nominal Rewriting Systems (NRS), ...

Overview: Theory of (In)Equational System

For (In)Equational Systems $\mathbb{S}$ on $\mathscr{D}$

- Construction of free models
- $\mathrm{Model}(\mathbb{S})$ is monadic over $\mathscr{D}$.
- $\mathrm{Model}(\mathbb{S})$ is cocomplete.



Motivation for definition: I. Signatures

Signatures as Endofunctors

- **Algebraic Theory / Term Rewriting System**

$\Sigma_{\mathsf{Num}} = \{\, \mathsf{zero} : 0, \mathsf{succ} : 1, \mathsf{plus} : 2 \,\}$

Σ_{Num}-algebra

- $D \in \mathbf{Set} \;/\; \mathbf{Pre}$
- $[\![\mathsf{zero}]\!] : D^0 \to D$
- $[\![\mathsf{succ}]\!] : D^1 \to D$
- $[\![\mathsf{plus}]\!] : D^2 \to D$

- **(In)Equational System**

$\Sigma_{\mathsf{Num}}(X) = X^0 + X^1 + X^2$ on $\mathbf{Set} \;/\; \mathbf{Pre}$

Σ_{Num}-algebra

- $D \in \mathbf{Set} \;/\; \mathbf{Pre}$
- $s : \Sigma_{\mathsf{Num}} D \to D$

  $: D^0 + D^1 + D^2 \to D$

Equations as parallel pairs of functors

$$\{\, x, y \,\} \vdash \mathsf{plus}(\mathsf{succ}(x), y) \;=\; \mathsf{succ}(\mathsf{plus}(x, y))$$

Algebraic Theory

$$\begin{array}{ccc} 1 & D & D^2 \\ {\scriptstyle [\![\mathsf{zero}]\!]}\downarrow & {\scriptstyle [\![\mathsf{succ}]\!]}\downarrow & {\scriptstyle [\![\mathsf{plus}]\!]}\downarrow \\ D & D & D \end{array} \;\longmapsto\; \forall \rho : \{\, x, y \,\} \to D \quad [\![\mathsf{plus}(\mathsf{succ}(x), y)]\!]_\rho \quad [\![\mathsf{succ}(\mathsf{plus}(x, y))]\!]_\rho \in D$$

Equational System

$$\boxed{\Sigma_{\mathsf{Num}}\text{-}\mathbf{Alg}} \;\rightrightarrows\; \boxed{(-)^2\text{-}\mathbf{Alg}}$$

$$\begin{array}{c} 1 + D + D^2 \\ {\scriptstyle [[\![\mathsf{zero}]\!],[\![\mathsf{succ}]\!],[\![\mathsf{plus}]\!]]}\downarrow \\ D \end{array} \;\longmapsto\; \begin{array}{ccc} & D^2 & \\ {\scriptstyle [\![\mathsf{succ}]\!] \times id} \swarrow & & \searrow {\scriptstyle [\![\mathsf{plus}]\!]} \\ D^2 & & D \\ {\scriptstyle [\![\mathsf{plus}]\!]} \searrow & & \swarrow {\scriptstyle [\![\mathsf{succ}]\!]} \\ & D & \end{array}$$

Motivation for definition: II. Equations

Equations as parallel pairs of functors

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Rewritings as parallel pairs of functors

$$\{\, x, y \,\} \vdash \mathsf{plus}(\mathsf{succ}(x), y) \;\longrightarrow\; \mathsf{succ}(\mathsf{plus}(x, y))$$

Term Rewriting System

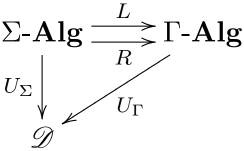
$$1 \xrightarrow{[\![\mathsf{zero}]\!]} D \qquad D \xrightarrow{[\![\mathsf{succ}]\!]} D \qquad D^2 \xrightarrow{[\![\mathsf{plus}]\!]} D \quad \longmapsto \quad \forall \rho : \{\, x, y \,\} \to D \quad [\![\mathsf{plus}(\mathsf{succ}(x), y)]\!]_\rho \leq [\![\mathsf{succ}(\mathsf{plus}(x, y))]\!]_\rho \in D$$

Inequational System

$$\boxed{\Sigma_{\mathsf{Num}}\text{-}\mathbf{Alg}} \;\rightrightarrows\; \boxed{(-)^2\text{-}\mathbf{Alg}}$$

$$1 + D + D^2 \xrightarrow{[[\![\mathsf{zero}]\!], [\![\mathsf{succ}]\!], [\![\mathsf{plus}]\!]]} D \quad \longmapsto \quad D^2 \xrightarrow{[\![\mathsf{succ}]\!] \times id} D^2 \xrightarrow{[\![\mathsf{plus}]\!]} D \;\;\leq\;\; D^2 \xrightarrow{[\![\mathsf{plus}]\!]} D \xrightarrow{[\![\mathsf{succ}]\!]} D$$

Definition of Equational System



- **Equational System** $\mathbb{T}$

$$(\mathscr{D} : \Sigma \rhd \Gamma \vdash L = R)$$

- **$\mathbb{T}$-Algebra**

$$(D, s : \Sigma D \to D)$$

such that

$$L(D, s) \;=\; R(D, s) : \Gamma D \to D$$

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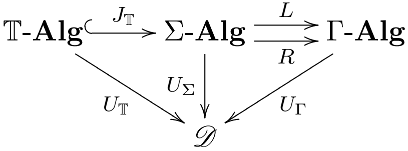

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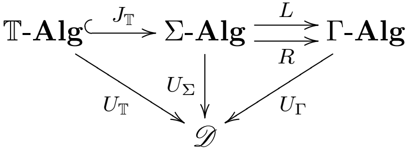


- **Inequational System** $\mathbb{T}$ **with** $\mathscr{D}$ **Pre-enriched**

$$(\mathscr{D} : \Sigma \rhd \Gamma \vdash L \leq R)$$

- $\mathbb{T}$**-Algebra**

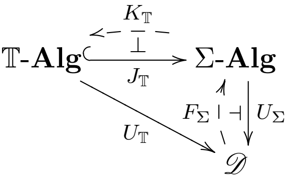
$$(D, s : \Sigma D \to D)$$

such that

$$L(D, s) \;\; \leq \;\; R(D, s) : \Gamma D \to D$$

Theorem: Basic Free Construction

For $\mathbb{T} = (\mathscr{D} : \Sigma \rhd \Gamma \vdash L \overset{\leq}{=} R)$ an (In)Equational System,



$\mathscr{D}$ is cocomplete.
Σ, Γ preserve ω-colimits.
(Σ, Γ preserve epimorphisms.)

- Construction of $F_{\Sigma}(V)$

$$0 \to V + \Sigma\, 0 \to V + \Sigma\,(V + \Sigma 0) \to \cdots \to (V + \Sigma(\text{-}))^{*} 0$$

- Construction of $K_{\mathbb{T}}(X, s : \Sigma X \to X)$

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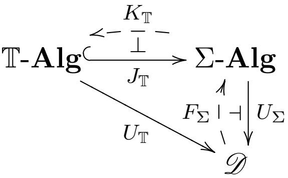

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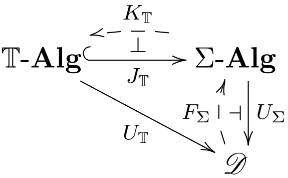

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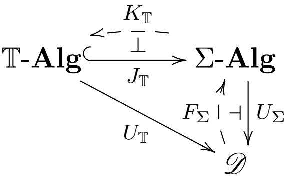

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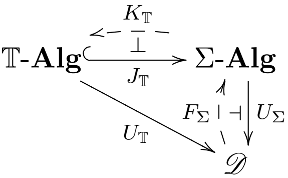

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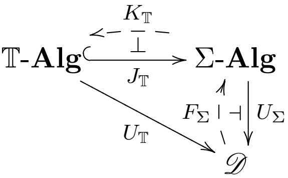

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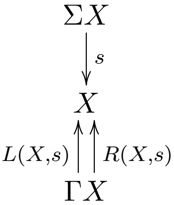

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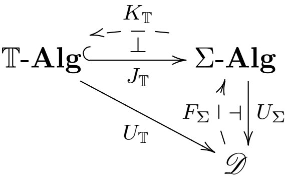

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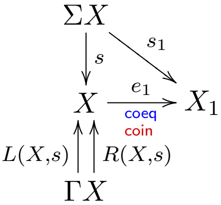

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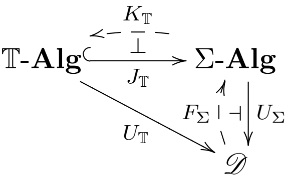

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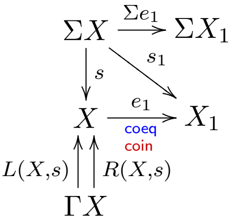

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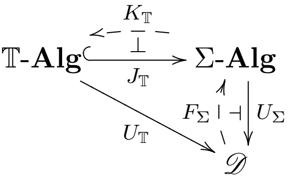

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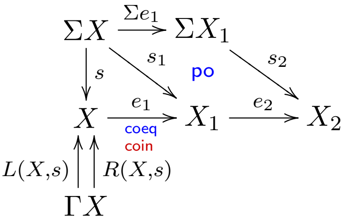

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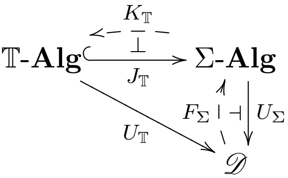

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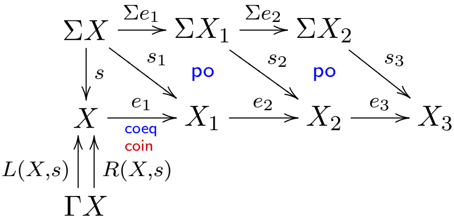

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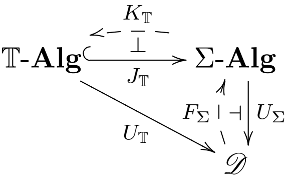

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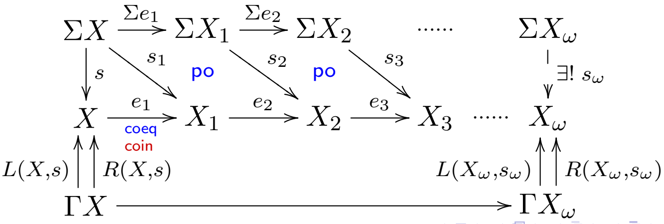

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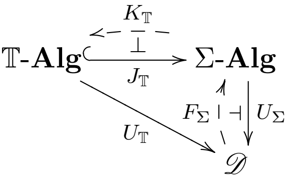

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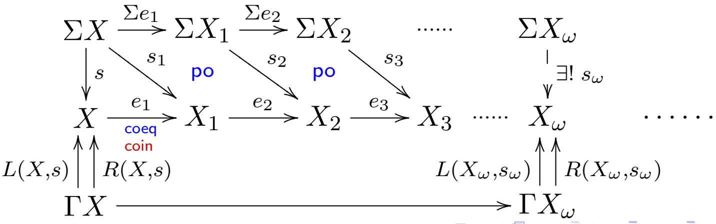

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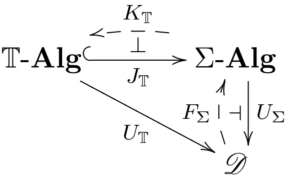

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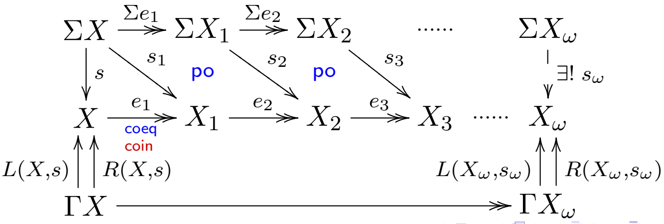
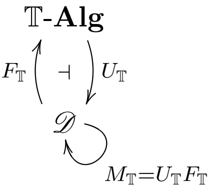


Theorem: Properties of Categories of Algebras

For $\mathbb{T} = (\mathscr{D} : \Sigma \rhd \Gamma \vdash L = R)$ an Equational System,



> $\mathscr{D}$ is cocomplete.
> Σ, Γ preserve ω-colimits.

1. $\mathbb{T}$-**Alg** is monadic over $\mathscr{D}$.
2. $\mathbb{T}$-**Alg** is cocomplete.

Overview: Nominal Equational Logic

Nominal Equational Logic (NEL) is an extension of first-order equational logic with names and assertions on their freshness.

Application of Equational Systems to NEL-theories provides

1. Cocompleteness and Monadicity of categories of algebras.
2. Explicit Construction of Free algebras.
   - Completeness of Nominal Equational Logic
   - Simplification of NEL-proofs

Nominal Equational Logic (Clouston & Pitts '07)

A NEL-theory $\mathbb{T} = (\Sigma, E)$

$\Sigma = \{\, \Sigma(n) \in \mathbf{Nom} \,\}_{n \in \mathbb{N}}$

$$t \;::=\; \sigma\, x \;\mid\; \mathsf{o}\; t_1 \ldots t_n$$

where $\sigma \in \mathsf{Perm}(\mathbb{A})$, $x \in \mathsf{V}$,
$\mathsf{o} \in \Sigma(n)$.

E is a set of axioms of the form

$A \blacktriangleright A_1 \# x_1, \ldots, A_n \# x_n$
$\vdash A' \# t = t'$
with
A_i's, $A', \mathsf{supp}(t), \mathsf{supp}(t') \subseteq A$

$\mathbb{T}_\lambda = (\Sigma_\lambda, E_\lambda)$

$$\Sigma_\lambda(0) = \{\, \langle a \rangle \mid a \in \mathbb{A} \,\} \cong \mathbb{A}$$
$$\Sigma_\lambda(1) = \{\, \lambda_a \mid a \in \mathbb{A} \,\} \cong \mathbb{A}$$
$$\Sigma_\lambda(2) = \{\, @ \,\} \cong 1$$

$a \blacktriangleright x \vdash a \# \lambda_a(x)$ (α)

$a \blacktriangleright a \# x \vdash \lambda_a(x @ \langle a \rangle) = x$ (η)

$a \blacktriangleright a \# x,\; y \vdash \lambda_a(x) @ y = x$ $(\beta\text{-}1)$

$\vdots$

Nominal Equational Logic (Clouston & Pitts '07)

A NEL-theory $\mathbb{T} = (\Sigma, E)$

$\Sigma = \{\, \Sigma(n) \in \mathbf{Nom} \,\}_{n \in \mathbb{N}}$

$$t \;::=\; \sigma\, x \;\mid\; \mathsf{o}\; t_1 \ldots t_n$$

where $\sigma \in \mathsf{Perm}(\mathbb{A})$, $x \in \mathsf{V}$, $\mathsf{o} \in \Sigma(n)$.

E is a set of axioms of the form

$A \blacktriangleright A_1 \# x_1, \ldots, A_n \# x_n$
$\vdash t = t'$

with
A_i's, $\mathsf{supp}(t), \mathsf{supp}(t') \subseteq A$

$\mathbb{T}_\lambda = (\Sigma_\lambda, E_\lambda)$

$$\Sigma_\lambda(0) = \{\, \langle a \rangle \mid a \in \mathbb{A} \,\} \cong \mathbb{A}$$
$$\Sigma_\lambda(1) = \{\, \lambda_a \mid a \in \mathbb{A} \,\} \cong \mathbb{A}$$
$$\Sigma_\lambda(2) = \{\, @ \,\} \cong 1$$

$a, b \blacktriangleright b\#x \vdash \lambda_a(x) = \lambda_b((a\,b)x)$ (α)

$a \blacktriangleright a\#x \vdash \lambda_a(x @ \langle a \rangle) = x$ (η)

$a \blacktriangleright a\#x,\; y \vdash \lambda_a(x) @ y = x$ $(\beta\text{-}1)$

$\vdots$

Nominal Equational Logic (Clouston & Pitts '07)

$\mathbb{T}$-Algebra (M, e)

$$M \in \mathbf{Nom}$$
$$\mathsf{e}_n : \Sigma(n) \times M^n \to M$$

satisfying all axioms

$$A \blacktriangleright \{ A_i \# x_i \}_{1 \le i \le n} \vdash t = t'$$

i.e.,

for all $\rho : \{ x_1, \ldots, x_n \} \to M$ s.t. $A_i \# \rho(x_i)$

$$\bar{\mathsf{e}}(t, \rho) = \bar{\mathsf{e}}(t, \rho)$$

$\mathbb{T}_\lambda$-Algebra (M, e)

$$M \in \mathbf{Nom}$$
$$\mathsf{e}_0 : \{ \langle a \rangle \mid a \in \mathbb{A} \} \times M^0 \to M$$
$$\mathsf{e}_1 : \{ \lambda_a \mid a \in \mathbb{A} \} \times M^1 \to M$$
$$\mathsf{e}_2 : \{ @ \} \times M^2 \to M$$

For the axiom (α), satisfying

$$a, b \blacktriangleright b \# x \vdash \lambda_a(x) = \lambda_b((a\, b)x)$$

i.e.,

for all $\rho : \{ x \} \to M$ s.t. $b \# \rho(x)$

$$\mathsf{e}_1(\lambda_a, \rho(x)) = \mathsf{e}_1(\lambda_b, (a\, b) \cdot \rho(x))$$

Nominal Equational Logic (Clouston & Pitts '07)

$\mathbb{T}$-Algebra (M, e)

$$M \in \mathbf{Nom}$$
$$\mathsf{e}_n : \Sigma(n) \times M^n \to M$$

satisfying for all $\alpha : A \rightarrowtail \mathbb{A}$

$$\alpha(A) \blacktriangleright \{\, \alpha(A_i) \# x_i \,\}_{1 \le i \le n}$$
$$\vdash \alpha(t) = \alpha(t')$$

Axioms are Invariant
under Renaming of atoms
by EQUIVARIANCE of e_n's

for all $\alpha : A \rightarrowtail \mathbb{A}$ and
for all $\rho : \{\, x_1, \ldots, x_n \,\} \to M$
s.t. $\alpha(A_i) \# \rho(x_i)$

$$\bar{\mathsf{e}}(\alpha(t), \rho) \;=\; \bar{\mathsf{e}}(\alpha(t'), \rho)$$

$\mathbb{T}_\lambda$-Algebra (M, e)

$$M \in \mathbf{Nom}$$
$$\mathsf{e}_0 : \{\, \langle a \rangle \mid a \in \mathbb{A} \,\} \times M^0 \to M$$
$$\mathsf{e}_1 : \{\, \lambda_a \mid a \in \mathbb{A} \,\} \times M^1 \to M$$
$$\mathsf{e}_2 : \{\, @ \,\} \times M^2 \to M$$

For the axiom (α),
satisfying for all $\alpha : \{\, a, b \,\} \rightarrowtail \mathbb{A}$

$$\alpha(a), \alpha(b) \blacktriangleright \alpha(b) \# x$$
$$\vdash \lambda_{\alpha(a)}(x) = \lambda_{\alpha(b)}((\alpha(a)\ \alpha(b))x)$$

i.e.,

for all $\alpha : \{\, a, b \,\} \rightarrowtail \mathbb{A}$ and
for all $\rho : \{\, x \,\} \to M$ s.t. $\alpha(b) \# \rho(x)$

$$\begin{aligned} & \mathsf{e}_1(\lambda_{\alpha(a)}, \rho(x)) \\ = \; & \mathsf{e}_1(\lambda_{\alpha(b)}, (\alpha(a)\ \alpha(b)) \cdot \rho(x)) \end{aligned}$$

NEL-Theories as Equational Systems

The Equational System $\overline{\mathbb{T}}$
for a NEL-theory $\mathbb{T} = (\Sigma, E)$

$$\overline{\mathbb{T}} = \mathbf{Nom} : \overline{\Sigma} \rhd \overline{\Gamma} \vdash \overline{L} = \overline{R}$$

$$\begin{aligned}
\overline{\Sigma} &= \textstyle\coprod_{n\in\mathbb{N}} \ \Sigma(n) \times (-)^n \\
\overline{\Gamma} &= \textstyle\coprod_{(\nabla\vdash t_1=t_2)\in E} \ \Gamma_\nabla \\
\overline{L} &= [\, F_{\nabla\vdash t_1} \,]_{(\nabla\vdash t_1=t_2)\in E} \\
\overline{R} &= [\, F_{\nabla\vdash t_2} \,]_{(\nabla\vdash t_1=t_2)\in E}
\end{aligned}$$

$$\Gamma_{A\blacktriangleright\{\, A_i\#x_i \,\}} \ (M) = \{\, \alpha : A \rightarrowtail \mathbb{A}, \rho : \{\, x_i \,\} \to M \mid \alpha(A_i)\#\rho(x_i) \,\}$$

$$F_{A\blacktriangleright\{\, A_i\#x_i \,\}\vdash t} \ (M, \mathsf{e}) \ (\alpha, \rho) = \overline{\mathsf{e}}(\alpha(t), \rho) \in M$$

$$\begin{aligned}
\overline{\Sigma_\lambda} \ (M) &= \{\, \langle a \rangle \mid a \in \mathbb{A} \,\} \times M^0 \\
&+ \ \{\, \lambda_a \mid a \in \mathbb{A} \,\} \times M^1 \\
&+ \ \{\, @ \,\} \times M^2
\end{aligned}$$

For the axiom (α)
$a, b \blacktriangleright b\#x \vdash \lambda_a(x) = \lambda_b((a\, b)x)$

$$\Gamma_{a,b\blacktriangleright b\#x} \ (M) = \{\, \alpha : \{\, a, b \,\} \rightarrowtail \mathbb{A}, \rho : \{\, x \,\} \to M \mid \alpha(b)\#\rho(x) \,\}$$

$$F_{a,b\blacktriangleright b\#x\vdash\lambda_a(x)} \ (M, \mathsf{e}) \ (\alpha, \rho) = \mathsf{e}_1(\lambda_{\alpha(a)}, \rho(x)) \in M$$

$$F_{a,b\blacktriangleright b\#x\vdash\lambda_b((a\,b)x)} \ (M, \mathsf{e}) \ (\alpha, \rho) = \mathsf{e}_1(\lambda_{\alpha(b)}, (\alpha(a)\ \alpha(b)) \cdot \rho(x)) \in M$$

Properties of Categories of Algebras for NEL-Theories

$\overline{\Sigma}, \overline{\Gamma}$ preserve filtered colimits and epimorphisms. (Easy!)

$$\mathbb{T}\text{-}\mathbf{Alg} \underset{F_{\mathbb{T}}}{\overset{U_{\mathbb{T}}}{\rightleftarrows}} \mathbf{Nom} \qquad F_{\mathbb{T}} \dashv U_{\mathbb{T}}$$

- $\overline{\mathbb{T}}\text{-}\mathbf{Alg}$ is monadic over $\mathbf{Nom}$.
- $\overline{\mathbb{T}}\text{-}\mathbf{Alg}$ is cocomplete.

Outline of Completeness Proof

For a NEL-theory $\mathbb{T}$

$$\text{(Goal)} \quad \nabla \vDash t = t' \implies \nabla \vdash t = t'$$

- Define a nominal set of terms $\mathbf{Tm}$ with syntactic structure e,
  a set of rules $\mathcal{R}_\approx$ generating $\approx$ on $\mathbf{Tm}$
  s.t. $(\mathbf{Tm}/_\approx, \mathsf{e}/_\approx)$ is a $\mathbb{T}$-algebra.
- $$\begin{array}{rl} & \nabla \vDash t = t' \\ \Rightarrow & \nabla \vDash_{(\mathbf{Tm}/_\approx, e/_\approx)} t = t' \\ \Rightarrow & [t]_\approx = [t']_\approx \in \mathbf{Tm}/_\approx \\ \Rightarrow & t \approx t' \text{ is provable by } \mathcal{R}_\approx \\ \Rightarrow & \nabla \vdash t = t' \text{ (by simulating } \mathcal{R}_\approx \text{ in } \mathbb{T}) \end{array}$$
-

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  s.t. $(\mathbf{Tm}/_{\approx}, \mathsf{e}/_{\approx})$ is a $\mathbb{T}$-algebra.
- $$\begin{aligned} & \nabla \vDash t = t' \\ \Rightarrow\ & \nabla \vDash_{(\mathbf{Tm}/_{\approx}, e/_{\approx})} t = t' \\ \Rightarrow\ & [\mathsf{enc}(\nabla \vdash t)]_{\approx} = [\mathsf{enc}(\nabla \vdash t')]_{\approx} \in \mathbf{Tm}/_{\approx} \\ \Rightarrow\ & \mathsf{enc}(\nabla \vdash t) \approx \mathsf{enc}(\nabla \vdash t') \text{ is provable by } \mathcal{R}_{\approx} \\ \Rightarrow\ & \nabla \vdash t = t' \text{ (by simulating } \mathcal{R}_{\approx} \text{ in } \mathbb{T}) \end{aligned}$$
-

Outline of Completeness Proof

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- $$\begin{array}{ll} & \nabla \vDash t = t' \\ \Rightarrow & \nabla \vDash_{(\mathbf{Tm}/_\approx, e/_\approx)} t = t' \\ \Rightarrow & [\mathsf{enc}(\nabla \vdash t)]_\approx = [\mathsf{enc}(\nabla \vdash t')]_\approx \ \in \mathbf{Tm}/_\approx \\ \Rightarrow & \mathsf{enc}(\nabla \vdash t) \approx \mathsf{enc}(\nabla \vdash t') \text{ is provable by } \mathcal{R}_\approx \\ \Rightarrow & \nabla \vdash t = t' \text{ (by simulating } \mathcal{R}_\approx \text{ in } \mathbb{T}) \end{array}$$
- $A \blacktriangleright \{ A_i \# x_i \} \vdash x_i \overset{\mathsf{enc}}{\Longrightarrow} x_i^{\mathsf{sort}(\overrightarrow{A - A_i})} \in \mathbf{Tm}$

Outline of Completeness Proof

For a NEL-theory $\mathbb{T}$

$$\text{(Goal)} \quad \nabla \vDash t = t' \implies \nabla \vdash t = t'$$

- Define a nominal set of terms $\mathbf{Tm}$ with syntactic structure e,
  a set of rules $\mathcal{R}_\approx$ generating $\approx$ on $\mathbf{Tm}$
  s.t. $(\mathbf{Tm}/_\approx, \mathsf{e}/_\approx)$ is a $\mathbb{T}$-algebra.
- $$\begin{aligned} & \nabla \vDash t = t' \\ \Rightarrow\ & \nabla \vDash_{(\mathbf{Tm}/_\approx, e/_\approx)} t = t' \\ \Rightarrow\ & [\mathsf{enc}(\nabla \vdash t)]_\approx = [\mathsf{enc}(\nabla \vdash t')]_\approx \ \in \mathbf{Tm}/_\approx \\ \Rightarrow\ & \mathsf{enc}(\nabla \vdash t) \approx \mathsf{enc}(\nabla \vdash t') \text{ is provable by } \mathcal{R}_\approx \\ \Rightarrow\ & \nabla \vdash t = t' \text{ (by simulating } \mathcal{R}_\approx \text{ in } \mathbb{T}) \end{aligned}$$
- $A \blacktriangleright \{ A_i \# x_i \} \vdash t \overset{\mathsf{enc}}{\Longrightarrow} t\{x_i^{\sigma \cdot \mathsf{sort}(\overrightarrow{A - A_i})} / \sigma x_i\} \in \mathbf{Tm}$
  e.g. $a, b \blacktriangleright b \# x \vdash \lambda_a(x) = \lambda_b((a\,b)x) \overset{\mathsf{enc}}{\Longrightarrow} \lambda_a(x^a) = \lambda_b(x^b)$

Construction of Term-Algebra

For a NEL-theory $\mathbb{T} = (\Sigma, E)$

For a countable set of variables V,

$$\widetilde{\mathsf{V}} \triangleq \mathsf{V} \times \mathsf{DList_{fin}}(\mathbb{A}) = \{ x^{\overrightarrow{A}} \mid x \in \mathsf{V},\ \overrightarrow{A} \in \mathsf{DList_{fin}}(\mathbb{A}) \}$$

From the construction

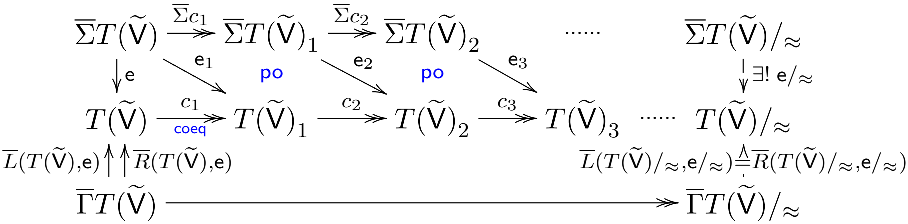
$$0 \to \widetilde{\mathsf{V}} + \overline{\Sigma}\,0 \to \widetilde{\mathsf{V}} + \overline{\Sigma}\,(\widetilde{\mathsf{V}} + \overline{\Sigma}0) \to \cdots \to T(\widetilde{\mathsf{V}})$$

A Free $\overline{\Sigma}$-algebra $(T(\widetilde{\mathsf{V}}), \mathsf{e})$ on $\widetilde{\mathsf{V}}$

$$\begin{array}{rcll} t \in T(\widetilde{\mathsf{V}}) & ::= & x^{\overrightarrow{A}} & (x \in \mathsf{V},\ \overrightarrow{A} \in \mathsf{DList_{fin}}(\mathbb{A})) \\ & | & \mathsf{o}\ t_1 \ldots t_n & (\mathsf{o} \in \Sigma(n)) \\ \mathsf{e}_n(\mathsf{o}, t_1, \ldots, t_n) & ::= & \mathsf{o}\ t_1 \ldots t_n & (\mathsf{o} \in \Sigma(n)) \end{array}$$

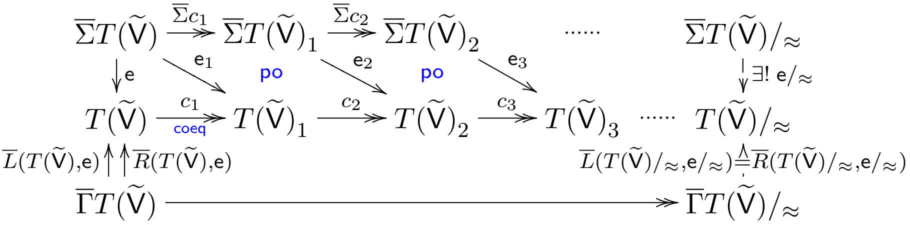
Construction of Term-Algebra

From the construction



Construction of Term-Algebra

From the construction



Rules $\mathcal{R}_\approx$

$$\mathsf{Ax}\ \frac{\alpha : A \rightarrowtail \mathbb{A},\ \{\, s_i \in T(\widetilde{\mathsf{V}}) \mid \alpha(A_i)\ \#\ \mathsf{supp}(s_i)\,\}}{\alpha(t)\{\sigma \cdot s_i/\sigma x_i\} \approx \alpha(t')\{\sigma \cdot s_i/\sigma x_i\}}\ (A \blacktriangleright \{\, A_i \# x_i \,\} \vdash t = t') \in E$$

$$\mathsf{Cong}\ \frac{\forall i \quad s_i \approx s_i'}{\mathsf{o}\ s_1 \ldots s_n \approx \mathsf{o}\ s_1' \ldots s_n'}\ \mathsf{o} \in \Sigma(n) \qquad \mathsf{Refl} \qquad \mathsf{Symm} \qquad \mathsf{Trans}$$

A Free $\overline{\mathbb{T}}$-algebra $(T(\widetilde{\mathsf{V}})/_\approx, \mathsf{e}/_\approx)$ on $(T(\widetilde{\mathsf{V}}), \mathsf{e})$

$$\mathsf{e}/_\approx(\mathsf{o}, [t_1]_\approx, \ldots, [t_n]_\approx) \ ::= \ [\mathsf{o}\ t_1 \ldots t_n]_\approx \quad (\mathsf{o} \in \Sigma(n))$$

Completeness of Nominal Equational Logic

Recall $\quad \mathsf{enc}(A \blacktriangleright \{\, A_i \# x_i \,\} \vdash t) \;\triangleq\; t[x_i^{\sigma\cdot\mathsf{sort}(\overrightarrow{A - A_i})} / \sigma x_i]$

$$A \blacktriangleright \{\, A_i \# x_i \,\} \vDash t = t'$$

$$\Rightarrow \quad A \blacktriangleright \{\, A_i \# x_i \,\} \vDash_{(T(\tilde{\mathsf{V}})/_\approx, e/_\approx)} t = t'$$

$$\Rightarrow \quad \overline{e/_\approx}(t, \rho : x_i \mapsto [x_i^{\mathsf{sort}(\overrightarrow{A - A_i})}]_\approx) = \overline{e/_\approx}(t', \rho : x_i \mapsto [x_i^{\mathsf{sort}(\overrightarrow{A - A_i})}]_\approx)$$

$$\Rightarrow \quad [\, t\{x_i^{\sigma\cdot\mathsf{sort}(\overrightarrow{A - A_i})} / \sigma x_i\} \,]_\approx = [\, t'\{x_i^{\sigma\cdot\mathsf{sort}(\overrightarrow{A - A_i})} / \sigma x_i\} \,]_\approx$$

$$\Rightarrow \quad \mathsf{enc}(A \blacktriangleright \{\, A_i \# x_i \,\} \vdash t) \;\approx\; \mathsf{enc}(A \blacktriangleright \{\, A_i \# x_i \,\} \vdash t')$$

By induction of the size of proof trees,

$\mathsf{enc}(\nabla \vdash t) \approx \mathsf{enc}(\nabla \vdash t')$ provable by $\mathcal{R}_\approx$
$\Longrightarrow \; \nabla \vdash t = t'$ provable in $\mathbb{T}$

Simplification of NEL Proofs

Rules $\mathcal{R}_{\mathbb{T}}$

$$\text{REFL}\ \frac{}{\nabla \vdash t = t} \qquad \text{SYMM}\ \frac{\nabla \vdash A \# t = t'}{\nabla \vdash A \# t' = t} \qquad \text{TRANS}\ \frac{\nabla \vdash A_1 \# t = t' \qquad \nabla \vdash A_2 \# t' = t''}{\nabla \vdash (A_1 \cup A_2) \# t = t''}$$

$$\text{SUBST}\ \frac{\nabla' \vdash \sigma = \sigma' : \nabla \qquad \nabla \vdash A \# t = t'}{\nabla' \vdash A \# t\{\sigma\} = t'\{\sigma'\}}\ \sigma, \sigma' \text{ substitutions}$$

$$\text{WEAK}\ \frac{\nabla \vdash A \# t = t'}{\nabla' \vdash A \# t = t'}\ \nabla \leq \nabla' \qquad \text{ATM-INTRO}\ \frac{\nabla \vdash A \# t = t'}{\nabla^{\# a} \vdash A \cup \{a\} \# t = t'}\ a \# (A, t, t')$$

$$\text{ATM-ELIM}\ \frac{\nabla^{\# a} \vdash A \# t = t'}{\nabla \vdash A \cup \{a\} \# t = t'}\ a \# (\nabla, A, t, t') \qquad \#\text{-EQUIVAR}\ \frac{}{A \blacktriangleright A_1 \# x \vdash \pi \cdot A_1 \# \pi x}$$

$$\text{SUSP}\ \frac{}{A \blacktriangleright \{a \mid \pi(a) \neq \pi'(a)\} \# x \vdash \pi \cdot x \# \pi' x} \qquad \text{AXIOM}\ \frac{}{\nabla \vdash A \# t = t'}\ (\nabla \vdash A \# t = t') \in E$$

Rules $\mathcal{R}_{\approx}$

$$\text{Ax}\ \frac{\alpha : A \rightarrowtail \mathbb{A},\ \{ s_i \mid \alpha(A_i) \# \mathsf{supp}(s_i) \}}{\alpha(t)\{\sigma \cdot s_i / \sigma x_i\} \approx \alpha(t')\{\sigma \cdot s_i / \sigma x_i\}}\ (A \blacktriangleright \{ A_i \# x_i \} \vdash t = t') \in E$$

$$\text{Cong}\ \frac{\forall i \quad s_i \approx s'_i}{\mathsf{o}\ s_1 \ldots s_n \approx \mathsf{o}\ s'_1 \ldots s'_n}\ \mathsf{o} \in \Sigma(n) \qquad \text{Refl} \qquad \text{Symm} \qquad \text{Trans}$$

$\nabla \vdash t = t'$ provable by $\mathcal{R}_{\mathbb{T}}$

$\Leftrightarrow$ $\mathsf{enc}(\nabla \vdash t) \approx \mathsf{enc}(\nabla \vdash t')$ provable by $\mathcal{R}_{\approx}$

$\Leftrightarrow$ $\mathsf{enc}(\nabla \vdash t) \xrightarrow{\text{Ax;Cong}*} s_1 \quad \cdots\cdots \quad s_{n-1} \xrightarrow{\text{Ax;Cong}*} \mathsf{enc}(\nabla \vdash t')$

Conclusion

1. Advantages of Equational Systems
   - More expressive than Enriched Algebraic Theories
   - Free theorems about Categories of Algebras
   - Explicit Construction of Free Algebras
2. Advantages of Inequational Systems
   - Semantic Models of Rewriting Systems.
   - Free Algebras are Syntactic Term Models.
   - Abstract Notions and Theorems can be developed.
     - Abstract Criteria for Modularity of Confluence, Termination

   In future
     - Abstract notion of Orthogonality.
     - Abstract Semantic Labeling.

       ...
3. Nominal Equational Logic (using Equational Systems)
   - Monadicity and Cocompleteness of Categories of Algebras
   - Completeness of Nominal Equational Logic
   - Simplification of Proofs