On the Construction of Free Algebras for Equational Systems

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Abstract

The purpose of this paper is threefold: to present a general abstract, yet practical, notion of equational system; to investigate and develop the finitary and transfinite construction of free algebras for equational systems; and to illustrate the use of equational systems as needed in modern applications.

Key words: Equational system; algebra; free construction; monad.

1 Introduction

The importance of equational theories in theoretical computer science is by now well established. Traditional applications include the initial algebra approach to the semantics of computational languages and the specification of abstract data types pioneered by the ADJ group [19], and the abstract description of powerdomain constructions as free algebras of non-determinism advocated by Plotkin [21,26] (see also [1]). While these developments essentially belong to the realm of universal algebra, more recent applications have had to be based on the more general categorical algebra. Examples include models of name-passing process calculi [12,32], theories of abstract syntax with variable binding [13,16], and the algebraic treatment of computational effects [27,28].

In the above and most other applications of equational theories, the existence and construction of initial and/or free algebras, and consequently of monads, plays a central role; as so does the study of categories of algebras. These topics are investigated here in the context of *equational systems*, a very broad

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notion of equational theory introduced by the authors in [11]. Examples of equational systems include enriched algebraic theories [23,31], algebras for a monad, monoids in a monoidal category, *etc.* (see Section 3).

The original motivation for the development of the theory of equational systems arose from the need of a mathematical theory readily applicable to two further examples of equational systems: (*i*) π-algebras (see Section 7.1), which provide algebraic models of the finitary π-calculus [32], and (*ii*) Σ-monoids (see Section 7.2), which are needed for the initial algebra approach to the semantics of languages with variable binding and capture-avoiding simultaneous substitution [13,10]. Indeed, these two examples respectively highlight two inadequacies of enriched algebraic theories in applications: (*i*) models may require a theory based on more than one enrichment, as it is the case with π-algebras; and (*ii*) the explicit presentation of an enriched algebraic theory may be hard to give, as it is the case with Σ-monoids.

Further benefits of equational systems over enriched algebraic theories are that the theory can be developed for cocomplete, not necessarily locally presentable, categories (examples of which are the category of topological spaces, the category of directed-complete posets, and the category of complete semilattices), and that the concept of equational system is straightforwardly dualizable: an *equational cosystem* on a category is simply an equational system on the opposite category (thus, for instance, comonoids in a monoidal category are algebras for an equational cosystem). On the other hand, the price paid for all this generality is that the important connection between enriched algebraic theories and enriched Lawvere theories [29] is lost for equational systems.

An equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ is defined as a parallel pair $L, R : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ of functors between categories of algebras over a base category $\mathscr{C}$. In this context, the endofunctor Σ on $\mathscr{C}$, which generalizes the notion of algebraic signature, is called a functorial signature; the functors L and R over $\mathscr{C}$ generalize the notion of equation and are called functorial terms; the endofunctor Γ on $\mathscr{C}$, referred to as a functorial context, corresponds to the context of the terms. The category of $\mathbb{S}$-algebras is the equalizer $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow \Sigma\text{-}\mathbf{Alg}$ of L, R. Thus, an $\mathbb{S}$-algebra is a Σ-algebra $(X, s : \Sigma X \to X)$ such that $L(X, s) = R(X, s)$ as Γ-algebras on X.

Free constructions for equational systems are investigated in Sections 4 and 5. For an equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$, the existence of free $\mathbb{S}$-algebras on objects in $\mathscr{C}$ is considered in two stages: (*i*) the construction of free Σ-algebras on objects in $\mathscr{C}$, and (*ii*) the construction of free $\mathbb{S}$-algebras over Σ-algebras. The former captures the construction of freely generated terms with operations from the functorial signature Σ; the latter that of quotienting Σ-algebras by the equation $L = R$ and congruence rules. We give finitary and transfinite sufficient conditions for the existence of free $\mathbb{S}$-algebras on Σ-algebras. The finitary condition can be used to deduce the existence of free algebras for enriched algebraic theories, but it applies more generally. The proofs of these results provide constructions of free algebras that may lead to explicit descriptions. Indeed, as concrete examples of this situation, we consider algebraic models of the untyped λ-calculus up to $\beta\eta$ identities (see Section 7.2) and the recently introduced nominal equational theories of Clouston and Pitts [7] (see Section 7.3).

Monads and categories of algebras for equational systems are discussed in Section 6. In the vein of the above results, we provide conditions under which the monadicity and cocompleteness of categories of algebras follow. As a direct application, we deduce that the categories of (*i*) π-algebras (Section 7.1), (*ii*) λ-algebras (Section 7.2), and (*iii*) algebras for nominal equational theories (Section 7.3) are monadic and cocomplete.

2 Algebraic equational theories

To set our work in context, we briefly review the classical concept of algebraic equational theory and some basic aspects of the surrounding theory (see *e.g.* [8]).

An algebraic equational theory consists of a signature defining its operations and a set of equations describing the axioms that it should obey.

A signature $\Sigma = (O, [-])$ is given by a set of operators O together with a function $[-] : O \to \mathbb{N}$ giving an arity to each operator. The set of terms $T_\Sigma(V)$ on a set of variables V is built up from the variables and the operators of the signature Σ by the following grammar

$$t \in T_\Sigma(V) \ ::= \ v \ | \ o(t_1, \ldots, t_k)$$

where $v \in V$, o is an operator of arity k, and $t_i \in T_\Sigma(V)$ for all $i = 1, \ldots, k$.

An equation of arity V for a signature Σ, written $\Sigma \rhd V \vdash l = r$, is given by a pair of terms $l, r \in T_\Sigma(V)$.

An algebraic equational theory $\mathbb{T} = (\Sigma, E)$ is given by a signature Σ together with a set of equations E.

An algebra for a signature Σ is a pair $(X, [\![-]\!]_X)$ consisting of a carrier set X together with interpretation functions $[\![o]\!]_X : X^{[o]} \to X$ for each operator o in Σ. By structural induction, such an algebra induces interpretations $[\![t]\!]_X : X^V \to X$ of terms $t \in T_\Sigma(V)$ as follows:

$$
[\![t]\!]_X = \begin{cases} X^V \xrightarrow{\pi_v} X & \text{, for } t = v \in V \\ X^V \xrightarrow{\langle [\![t_1]\!]_X, \ldots, [\![t_k]\!]_X \rangle} X^k \xrightarrow{[\![o]\!]_X} X & \text{, for } t = o(t_1, \ldots, t_k) \end{cases}
$$

An algebra for the theory $\mathbb{T} = (\Sigma, E)$ is an algebra for the signature Σ that satisfies the constraints given by the equations in E, where a Σ-algebra $(X, [\![-]\!]_X)$ is said to satisfy the equation $\Sigma \rhd V \vdash l = r$ whenever $[\![l]\!]_X \vec{x} = [\![r]\!]_X \vec{x}$ for all $\vec{x} \in X^V$.

A homomorphism of $\mathbb{T}$-algebras from $(X, [\![-]\!]_X)$ to $(Y, [\![-]\!]_Y)$ is a function $h : X \to Y$ between their carrier sets that commutes with the interpretation of each operator; that is, such that $h([\![o]\!]_X(x_1, \ldots, x_k)) = [\![o]\!]_Y(h(x_1), \ldots, h(x_k))$ for all $x_i \in X$. Algebras and homomorphisms form the category $\mathbb{T}$-**Alg**.

The existence of free algebras for algebraic theories is one of the most significant properties that they enjoy. For an algebraic theory $\mathbb{T} = (\Sigma, E)$, the free algebra over a set X has as carrier the set $T_\Sigma(X)/_{\approx_E}$ of equivalence classes of terms on X under the equivalence relation $\approx_E$ defined by setting $t \approx_E t'$ iff t is provably equal to t' by the equations given in E and the congruence rules. The interpretation of each operator on $T_\Sigma(X)/_{\approx_E}$ is given syntactically: $[\![o]\!]([t_1]_{\approx_E}, \ldots, [t_k]_{\approx_E}) = [o(t_1, \ldots, t_k)]_{\approx_E}$. This construction gives rise to a left adjoint to the forgetful functor $U_\mathbb{T} : \mathbb{T}\text{-}\mathbf{Alg} \to \mathbf{Set}$. Moreover, the adjunction is monadic: $\mathbb{T}$-**Alg** is equivalent to the category of algebras for the associated monad on **Set**.

3 Equational systems

We develop abstract notions of signature and equation, leading to the concept of equational system. Free constructions for equational systems are considered in the following two sections.

3.1 Functorial signatures

We recall the notion of algebra for an endofunctor and how it generalizes that of algebra for a signature.

An algebra for an endofunctor Σ on a category $\mathscr{C}$ is a pair (X, s) consisting of a carrier object X in $\mathscr{C}$ together with an algebra structure map $s : \Sigma X \to X$. A homomorphism of Σ-algebras $(X, s) \to (Y, t)$ is a map $h : X \to Y$ in $\mathscr{C}$ such that $h \circ s = t \circ \Sigma h$. Σ-algebras and homomorphisms form the category

Σ-**Alg**, and the forgetful functor $U_\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ maps a Σ-algebra (X, s) to its carrier object X.

As it is well known, every algebraic signature can be turned into an endofunctor on **Set** preserving its algebras. Indeed, for a signature Σ, one defines the corresponding endofunctor as $\overline{\Sigma}(X) = \coprod_{o\in\Sigma} X^{[o]}$, so that Σ-**Alg** and $\overline{\Sigma}$-**Alg** are isomorphic. In this view, we will henceforth take endofunctors as a general abstract notion of signature.

Definition 3.1 (Functorial signature) *A functorial signature on a category is an endofunctor on it.*

3.2 Functorial terms

We motivate and present an abstract notion of term for functorial signatures.

Let $t \in T_\Sigma(V)$ be a term on a set of variables V for a signature Σ. Recall from the previous section that for every Σ-algebra $(X, [\![-]\!]_X)$, the term t gives an interpretation function $[\![t]\!]_X : X^V \to X$. Thus, writing Γ_V for the endofunctor $(-)^V$ on **Set**, the term t determines a function $\bar{t}$ assigning to a Σ-algebra $(X, [\![-]\!]_X)$ the Γ_V-algebra $(X, [\![t]\!]_X)$. Note that the function $\bar{t}$ does not only preserve carrier objects but, furthermore, by the uniformity of the interpretation of terms, that a Σ-homomorphism $(X, [\![-]\!]_X) \to (Y, [\![-]\!]_Y)$ is also a Γ_V-homomorphism $(X, [\![t]\!]_X) \to (Y, [\![t]\!]_Y)$. In other words, the function $\bar{t}$ extends to a functor $\Sigma\text{-}\mathbf{Alg} \to \Gamma_V\text{-}\mathbf{Alg}$ over **Set** (*i.e.* a functor preserving carrier objects and homomorphisms). These considerations lead us to define an abstract notion of term in context as follows.

Definition 3.2 (Functorial term) *Let Σ be a functorial signature on a category $\mathscr{C}$. A functorial term $\mathscr{C} : \Sigma \rhd \Gamma \vdash T$ consists of an endofunctor Γ on $\mathscr{C}$, referred to as a functorial context, and a functor $T : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ over $\mathscr{C}$; that is, a functor such that $U_\Gamma \circ T = U_\Sigma$.*

Typically, when a syntactic signature Σ is turned into a functorial signature $\overline{\Sigma}$ its algebras provide the models of the signature, giving interpretations to the operators. Moreover, when a syntactic term in context $\Gamma \vdash t$ is turned into a functorial term $\bar{t} : \overline{\Sigma}\text{-}\mathbf{Alg} \to \overline{\Gamma}\text{-}\mathbf{Alg}$, the object $\overline{\Gamma}X$ intuitively consists of all valuations of the context Γ in X, and the functor $\bar{t}$ encodes the process of evaluating a term to a value, parametrically on models and valuations.

We give a general example of functorial term that arises frequently in applications. To this end, let T_Σ be the free monad on a functorial signature Σ on a category $\mathscr{C}$. For an object $V \in \mathscr{C}$, to be thought of as an object of variables, the object $T_\Sigma V$ intuitively represents the terms built up from the variables by means of the signature. Under this view, thus, we obtain an abstract notion of term as a generalized element $U \to T_\Sigma V$. Assume now that $\mathscr{C}$ is symmetric monoidal closed (with structure I, $\otimes$, $[-,=]$) and that Σ is strong [24], with strength $\mathsf{st}_{X,V} : X \otimes \Sigma(V) \to \Sigma(X \otimes V)$. It follows that T_Σ is strong, say with strength $\overline{\mathsf{st}}_{X,V} : X \otimes T_\Sigma(V) \to T_\Sigma(X \otimes V)$ providing a means to distribute parameters within terms as specified by st. In this situation, then, every abstract term $t : U \to T_\Sigma V$ induces a functorial term $\bar{t} : \Sigma\text{-}\mathbf{Alg} \to \Gamma_{U,V}\text{-}\mathbf{Alg}$, for the functorial context $\Gamma_{U,V}(-) = [V,-] \otimes U$, as follows:

$$\begin{aligned}&\bar{t}(X, s : \Sigma X \to X)\\ &\quad = \big(\, [V,X] \otimes U \xrightarrow{\mathrm{id} \otimes t} [V,X] \otimes T_\Sigma(V) \xrightarrow{\overline{\mathsf{st}}} T_\Sigma([V,X] \otimes V) \xrightarrow{T_\Sigma(ev)} T_\Sigma X \xrightarrow{\bar{s}} X \,\big)\end{aligned}$$

where $(X, \bar{s} : T_\Sigma X \to X)$ is the T_Σ-algebra corresponding to the Σ-algebra (X, s).

3.3 Equational systems

We define equational systems, our abstract notion of equational theory.

Definition 3.3 (Equational system) *An equational system*

$$\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$$

is given by a functorial signature Σ on a category $\mathscr{C}$, and a pair of functorial terms $\mathscr{C} : \Sigma \rhd \Gamma \vdash L$ and $\mathscr{C} : \Sigma \rhd \Gamma \vdash R$ referred to as a functorial equation.

We have restricted attention to equational systems subject to a single equation. The consideration of multi-equational systems $(\mathscr{C} : \Sigma \rhd \{\, \Gamma_i \vdash L_i = R_i \,\}_{i \in I})$ subject to a set of equations in what follows is left to the interested reader. We remark however that our development is typically without loss of generality; as, whenever $\mathscr{C}$ has I-indexed coproducts, a multi-equational system as above can be expressed as the equational system $(\mathscr{C} : \Sigma \rhd \coprod_{i\in I} \Gamma_i \vdash [L_i]_{i \in I} = [R_i]_{i \in I})$ with a single equation.

Recall that an equation $\Sigma \rhd V \vdash l = r$ in an algebraic theory is interpreted as the constraint that the interpretation functions associated with the terms l and r coincide. Hence, for an equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$, it is natural to say that a Σ-algebra (X, s) satisfies the functorial equation $\Gamma \vdash L = R$ whenever $L(X, s) = R(X, s) : \Gamma X \to X$, and consequently to define the category of algebras for the equational system as the full subcategory of Σ-$\mathbf{Alg}$ consisting of the Σ-algebras that satisfy the functorial equation $\Gamma \vdash L = R$. Equivalently, we introduce the following definition.

Definition 3.4 *For an equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$, the category $\mathbb{S}$-$\mathbf{Alg}$ of $\mathbb{S}$-algebras is the equalizer of $L, R : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ (in the large category of locally small categories over $\mathscr{C}$).*

Examples of equational systems together with their induced categories of algebras follow.

(1) The equational system $\mathbb{S}_\mathbb{T}$ associated to an algebraic theory $\mathbb{T} = (\Sigma, E)$ is given by $(\mathbf{Set} : \Sigma_\mathbb{T} \rhd \Gamma_\mathbb{T} \vdash L_\mathbb{T} = R_\mathbb{T})$, with $\Sigma_\mathbb{T} X = \coprod_{o\in\Sigma} X^{[o]}$, $\Gamma_\mathbb{T} X = \coprod_{(V\vdash l=r)\in E} X^V$, and

$$L_\mathbb{T}(X, [\![-]\!]_X) = \Big(X, \big[\,[\![l]\!]_X\,\big]_{(V\vdash l=r)\in E}\Big) \ ,$$
$$R_\mathbb{T}(X, [\![-]\!]_X) = \Big(X, \big[\,[\![r]\!]_X\,\big]_{(V\vdash l=r)\in E}\Big) \ .$$

It follows that $\mathbb{T}$-**Alg** is isomorphic to $\mathbb{S}_\mathbb{T}$-**Alg**.

(2) More generally, consider an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, B, E, \sigma, \tau)$ on a locally finitely presentable category $\mathscr{C}$ enriched over a suitable category $\mathscr{V}$, see [23]. Recall that this is given by functors $B, E : |\mathscr{C}_{\mathrm{fp}}| \to \mathscr{C}_0$ and a pair of morphisms $\sigma, \tau : FE \to FB$ between the free finitary monads FB and FE on $\mathscr{C}$ respectively arising from B and E. The equational system $\mathbb{S}_\mathbb{T}$ associated to such an enriched algebraic theory $\mathbb{T}$ is given by $(\mathscr{C}_0 : (GB)_0 \rhd (GE)_0 \vdash \overline{\sigma}_0 = \overline{\tau}_0)$, where GB and GE are the free finitary endofunctors on $\mathscr{C}$ respectively arising from B and E, and where $\overline{\sigma}$ and $\overline{\tau}$ are respectively the functors corresponding to σ and τ by the bijection between morphisms $FE \to FB$ and functors $GB\text{-}\mathbf{Alg} \cong \mathscr{C}^{FB} \to \mathscr{C}^{FE} \cong GE\text{-}\mathbf{Alg}$ over $\mathscr{C}$. It follows that $(\mathbb{T}\text{-}\mathbf{Alg})_0$ is isomorphic to $\mathbb{S}_\mathbb{T}$-**Alg**.

(3) The definition of Eilenberg-Moore algebra for a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ with binary coproducts can be directly encoded as the equational system $\mathbb{S}_\mathbf{T} = (\mathscr{C} : T \rhd \Gamma \vdash L = R)$ with $\Gamma(X) = X + T^2X$ and

$$L(X, s) = (\,X, [\ s \circ \eta_X\ ,\ s \circ \mu_X\ ]\,)\ ,$$
$$R(X, s) = (\,X, [\ \ \mathrm{id}_X\ \ ,\ s \circ Ts\ ]\,)\ .$$

It follows that $\mathbb{S}_\mathbf{T}$-**Alg** is isomorphic to the category $\mathscr{C}^\mathbf{T}$ of Eilenberg-Moore algebras for $\mathbf{T}$.

(4) The definition of monoid in a monoidal category $(\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ with binary coproducts yields the equational system

$$\mathbb{S}_{\mathrm{Mon}(\mathscr{C})} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$$

with $\Sigma(X) = (X \otimes X) + I$, $\Gamma(X) = \big((X \otimes X) \otimes X\big) + (I \otimes X) + (X \otimes I)$, and

$$L(X, [m, e]) = (\,X, [\quad m \circ (m \otimes \mathrm{id}_X)\quad ,\quad \lambda_X\quad ,\quad \rho_X\quad]\,)\,,$$
$$R(X, [m, e]) = (\,X, [\ m \circ (\mathrm{id}_X \otimes m) \circ \alpha_{X,X,X}\ , m \circ (e \otimes \mathrm{id}_X)\ , m \circ (\mathrm{id}_X \otimes e)\ ]\,)\,.$$

It follows that $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$-**Alg** is isomorphic to the category of monoids and monoid homomorphisms in $\mathscr{C}$.

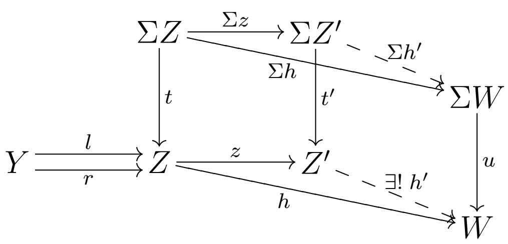
4 Finitary free constructions for equational systems

We give sufficient finitary conditions for the existence of free algebras for equational systems; that is, for the existence of a left adjoint to the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$, for $\mathbb{S}$ an equational system. Since, by definition, the forgetful functor $U_{\mathbb{S}}$ decomposes as $\mathbb{S}\text{-}\mathbf{Alg} \xhookrightarrow{J_{\mathbb{S}}} \Sigma\text{-}\mathbf{Alg} \xrightarrow{U_\Sigma} \mathscr{C}$, its left adjoint can be described in two stages as the composition of a left adjoint to U_Σ followed by a left adjoint to $J_{\mathbb{S}}$. Conditions for the existence of the former have been studied in the literature (see *e.g.* [3,4]). Thus, we concentrate here on obtaining a reflection to the embedding of $\mathbb{S}\text{-}\mathbf{Alg}$ into $\Sigma\text{-}\mathbf{Alg}$.

4.1 Algebraic coequalizers

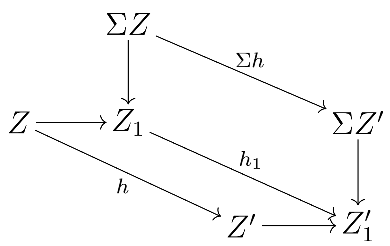
The construction of free algebras for an equational system explained in Section 4.3 depends on the key concept of *algebraic coequalizer*, whose existence and explicit construction is dealt with in here.

Definition 4.1 *Let Σ be an endofunctor on a category $\mathscr{C}$. By a Σ-algebraic coequalizer of a parallel pair l, r in $\mathscr{C}$ into the carrier object Z of a Σ-algebra (Z, t) we mean a universal Σ-algebra homomorphism z from (Z, t) coequalizing the parallel pair.*



The following lemma, shows how algebraic coequalizers may be seen to arise from coequalizers by *reflecting algebra cospans* to algebras.

Definition 4.2 *For Σ an endofunctor on a category $\mathscr{C}$, we let Σ-*$\mathbf{AlgCoSpan}$ *be the category with Σ-algebra cospans $(Z \to Z_1 \leftarrow \Sigma Z)$ as objects and homomorphisms (h, h_1) between them as follows:*



We will henceforth regard Σ-**Alg** as a full subcategory of Σ-**AlgCoSpan** via the embedding that maps $(Z \leftarrow \Sigma Z)$ to $(Z \stackrel{\mathrm{id}}{\to} Z \leftarrow \Sigma Z)$.

Lemma 4.3 *Let Σ be an endofunctor on a category $\mathscr{C}$. If the embedding Σ-**Alg** $\hookrightarrow$ Σ-**AlgCoSpan** has a left adjoint, then the existence of coequalizers in $\mathscr{C}$ implies that of Σ-algebraic coequalizers.*

PROOF.

$$\begin{array}{ccccccc}
 & & \Sigma Z & & \xrightarrow{\quad\Sigma z\quad} & & \Sigma Z' \\
 & & \downarrow{\scriptstyle t} & \searrow^{c\circ t} & \textsf{reflect} & & \downarrow{\scriptstyle t'} \\
Y & \underset{r}{\overset{l}{\rightrightarrows}} & Z & \underset{\textsf{coeq}}{\overset{c}{\twoheadrightarrow}} & Z_1 & \xrightarrow{z_1} & Z'
\end{array}$$

Let l, r be a parallel pair into Z in $\mathscr{C}$ and let $t : \Sigma Z \to Z$ be an algebra structure. Consider the coequalizer $c : Z \twoheadrightarrow Z_1$ of l, r in $\mathscr{C}$ and let $(z, z_1) : (Z \stackrel{c}{\to} Z_1 \stackrel{c\circ t}{\leftarrow} \Sigma Z) \longrightarrow (Z' \stackrel{\mathrm{id}}{\to} Z' \stackrel{t'}{\leftarrow} \Sigma Z')$ be a universal reflection. Then, the homomorphism $z = z_1 \circ c : (Z, t) \to (Z', t')$ is a Σ-algebraic coequalizer of l, r.

$\square$

The missing ingredient for constructing algebraic coequalizers, thus, is the construction of a reflection from Σ-**AlgCoSpan** to Σ-**Alg**.

Theorem 4.4 *Let Σ be an endofunctor on a category $\mathscr{C}$ with finite colimits. If $\mathscr{C}$ has colimits of ω-chains and Σ preserves them then Σ-**Alg** is a full reflective subcategory of Σ-**AlgCoSpan**.*

PROOF. Given a Σ-algebra cospan $(c_0 : Z_0 \to Z_1 \leftarrow \Sigma Z_0 : t_0)$ we construct a Σ-algebra $t_\infty : \Sigma Z_\infty \to Z_\infty$ as follows:

$$\begin{array}{cccccccccc}
\Sigma Z_0 & \xrightarrow{\Sigma c_0} & \Sigma Z_1 & \xrightarrow{\Sigma c_1} & \Sigma Z_2 & \xrightarrow{\Sigma c_2} & \Sigma Z_3 & \cdots & \Sigma Z_\infty & \\
 & \searrow^{t_0} \;\textsf{po} & & \searrow^{t_1} \;\textsf{po} & & \searrow^{t_2} & & & \downarrow{\scriptstyle \exists!\, t_\infty} & \\
Z_0 & \xrightarrow[c_0]{} & Z_1 & \xrightarrow[c_1]{} & Z_2 & \xrightarrow[c_2]{} & Z_3 & \cdots & Z_\infty & \textsf{colim}
\end{array} \qquad (1)$$

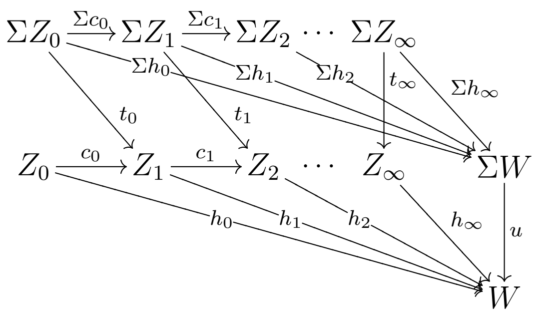
where

- $Z_{n+1} \stackrel{c_{n+1}}{\longrightarrow} Z_{n+2} \stackrel{t_{n+1}}{\longleftarrow} \Sigma Z_{n+1}$ is a pushout of $Z_{n+1} \stackrel{t_n}{\longleftarrow} \Sigma Z_n \stackrel{\Sigma c_n}{\longrightarrow} \Sigma Z_{n+1}$, for all $n \geq 0$;
- Z_∞ with $\{\, \overline{c}_n : Z_n \to Z_\infty \,\}_{n\geq 0}$ is a colimit of the ω-chain $\{\, c_n \,\}_{n\geq 0}$; and
- t_∞ is the mediating map from the colimiting cone $\{\, \Sigma\overline{c}_n : \Sigma Z_n \to \Sigma Z_\infty \,\}_{n\geq 0}$ to the cone $\{\, \overline{c}_{n+1} \circ t_n \,\}_{n\geq 0}$ of the ω-chain $\{\, \Sigma c_n \,\}_{n\geq 0}$.

We now show that the homomorphism

$$(\overline{c}_0, \overline{c}_1) : (Z_0 \to Z_1 \leftarrow \Sigma Z_0) \longrightarrow (Z_\infty \stackrel{\mathrm{id}}{\to} Z_\infty \leftarrow \Sigma Z_\infty)$$

in Σ-**AlgCoSpan** is universal. To this end, consider another homomorphism $(h_0, h_1) : (Z_0 \to Z_1 \leftarrow \Sigma Z_0) \longrightarrow (W \xrightarrow{\text{id}} W \xleftarrow{u} \Sigma W)$ and perform the following construction



where

- for $n \geq 0$, h_{n+2} is the mediating map from the pushout Z_{n+2} to W with respect to the cone $(h_{n+1} : Z_{n+1} \to W \leftarrow \Sigma Z_{n+1} : u \circ \Sigma h_{n+1})$; and
- h_∞ is the mediating map from the colimit Z_∞ to W with respect to the cone $\{ h_n \}_{n \geq 0}$ of the ω-chain $\{ c_n \}_{n \geq 0}$.

As, for all $n \geq 0$, $u \circ \Sigma h_\infty \circ \Sigma \overline{c}_n = h_\infty \circ t_\infty \circ \Sigma \overline{c}_n$, it follows that h_∞ is a Σ-algebra homomorphism. Hence, (h_0, h_1) factors as $(h_\infty, h_\infty) \circ (\overline{c}_0, \overline{c}_1)$.

We finally establish the uniqueness of such factorizations. Indeed, for any homomorphism $h : (Z_\infty, t_\infty) \to (W, u)$ such that $h \circ \overline{c}_1 = h_1$, it follows by induction that $h \circ \overline{c}_n = h_n$ for all $n \geq 0$, and hence that $h = h_\infty$. $\square$

Corollary 4.5 *Let Σ be an endofunctor on a category $\mathscr{C}$ with finite colimits. If $\mathscr{C}$ has colimits of ω-chains and Σ preserves them then Σ-algebraic coequalizers exist. If, in addition, Σ preserves epimorphisms then Σ-algebraic coequalizers are epimorphic in $\mathscr{C}$.*

PROOF. According to Lemma 4.3 and Theorem 4.4, the algebraic coequalizer of $l, r : Y \to Z_0$ with respect to the algebra structure $t : \Sigma Z_0 \to Z_0$ is given by $\overline{c}_0 : (Z_0, t) \to (Z_\infty, t_\infty)$ in the construction (1) above where c_0 is taken to be the coequalizer of l, r in $\mathscr{C}$ and t_0 is defined as $c_0 \circ t$.

If Σ preserves epimorphisms, then the ω-chain $\{ c_n : Z_n \to Z_{n+1} \}_{n \geq 0}$ in (1) consists of epimorphisms, and hence this is also the case for its colimiting cone $\{ \overline{c}_n : Z_n \to Z_\infty \}_{n \geq 0}$. $\square$

The following result describes a well-known condition for the existence of free Σ-algebras (see *e.g.* [2]).

Theorem 4.6 *Let* Σ *be an endofunctor on a category* $\mathscr{C}$ *with finite coproducts. If* $\mathscr{C}$ *has colimits of* ω*-chains and* Σ *preserves them, then the forgetful functor* $U_\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ *has a left adjoint.*

The free Σ-algebra $(TX, \tau_X : \Sigma(TX) \to TX)$ on an object $X \in \mathscr{C}$ and the unit map $\eta_X : X \to TX$ are constructed as follows. The object TX is given as a colimit of the ω-chain $\{ f_n : X_n \to X_{n+1} \}_{n \geq 0}$, inductively defined by $X_0 = 0$, $f_0 = \,!$ and $X_{n+1} = X + \Sigma X_n$, $f_{n+1} = X + \Sigma f_n$ for $n \geq 0$, where 0 is an initial object and ! is the unique map. Since the functor $X + \Sigma(-)$ preserves colimits of ω-chains, the object $X + \Sigma(TX)$ is a colimit of the ω-chain $\{ X + \Sigma f_n : X + \Sigma X_n \to X + \Sigma X_{n+1} \}_{n \geq 0}$. The map $[\eta_X, \tau_X]$ is the unique mediating map as follows:

$$
\begin{array}{ccccccccc}
X + \Sigma 0 & \xrightarrow{X+\Sigma !} & X + \Sigma(X + \Sigma 0) & & \cdots\cdots & & & & X + \Sigma(TX) \\
& \searrow^{=} & & \searrow^{=} & & & & & \downarrow{\scriptstyle \cong \, \exists! \, [\eta_X, \tau_X]} \\
0 & \xrightarrow{!} & X + \Sigma 0 & \xrightarrow{X+\Sigma !} & X + \Sigma(X + \Sigma 0) & & \cdots\cdots & & TX \ \text{colim}
\end{array}
\qquad (2)
$$

The intuition behind this construction of TX, in which Σ represents a signature and X an object of variables, is that of taking the union of the sequence of objects X_n of terms of depth at most n built from the operators in Σ and variables in X.

Note that the $(X + \Sigma(-))$-algebra in the construction (2) is obtained as the reflection of the initial $(X + \Sigma(-))$-algebra cospan $(0 \xrightarrow{!} X + \Sigma 0 \xleftarrow{\text{id}} X + \Sigma 0)$ as given by the construction (1).

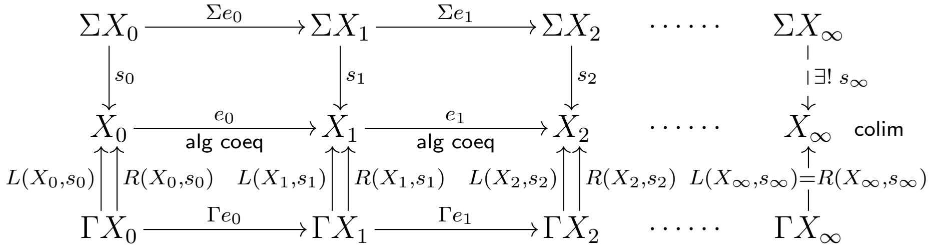
4.3 Finitary free $\mathbb{S}$*-algebras*

We now turn our attention to finitary conditions for the existence of a left adjoint to the embedding $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow \Sigma\text{-}\mathbf{Alg}$. The construction of free $\mathbb{S}$-algebras on Σ-algebras follows.

Theorem 4.7 *Let* $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ *be an equational system for which* $\mathscr{C}$ *is finitely cocomplete. If* $\mathscr{C}$ *has colimits of* ω*-chains, and* Σ *and* Γ *preserve them then* $\mathbb{S}\text{-}\mathbf{Alg}$ *is a full reflective subcategory of* $\Sigma\text{-}\mathbf{Alg}$.

This result is proved by performing an iterative construction that associates a free $\mathbb{S}$-algebra to every Σ-algebra. The cocompleteness assumptions on the base category allows one to perform the construction, while the other conditions guarantee that the process stops.

PROOF. Given a Σ-algebra (X_0, s_0), we construct a free $\mathbb{S}$-algebra (X_∞, s_∞) on it as follows:



(3)

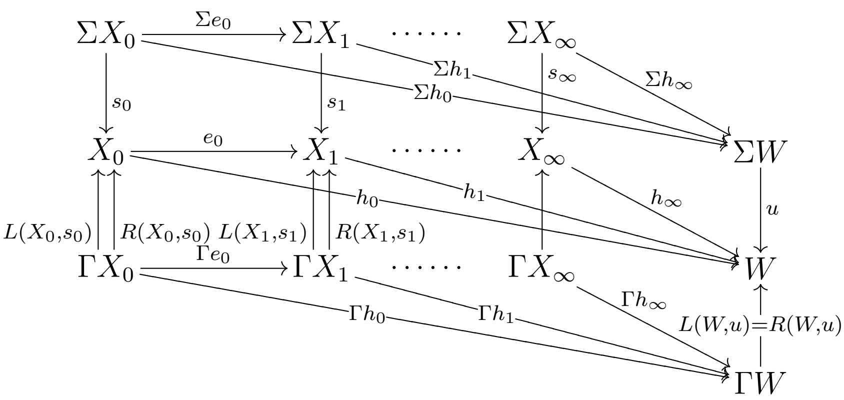
where

- for $n \geq 0$, $e_n : (X_n, s_n) \to (X_{n+1}, s_{n+1})$ is an algebraic coequalizer of the parallel pair $L(X_n, s_n), R(X_n, s_n) : \Gamma X_n \to X_n$;
- X_∞ with $\{\, \overline{e}_n : X_n \to X_\infty \,\}_{n\geq 0}$ is a colimit of the ω-chain $\{\, e_n \,\}_{n\geq 0}$; and
- s_∞ is the mediating map from the colimiting cone $\{\, \Sigma\overline{e}_n \,\}_{n\geq 0}$ to the cone $\{\, \overline{e}_n \circ s_n \,\}_{n\geq 0}$ of the ω-chain $\{\, \Sigma e_n \,\}_{n\geq 0}$.

As $\{\, \Gamma\overline{e}_n \,\}_{n\geq 0}$ is a colimiting cone and $L(X_\infty, s_\infty) \circ \Gamma\overline{e}_n = R(X_\infty, s_\infty) \circ \Gamma\overline{e}_n$ for all $n \geq 0$, it follows that (X_∞, s_∞) is an $\mathbb{S}$-algebra.

We now show that the unit $\eta = \overline{e}_0 : (X_0, s_0) \to (X_\infty, s_\infty)$ satisfies the universal property that every homomorphism $(X_0, s_0) \to (W, u)$ into an $\mathbb{S}$-algebra (W, u) uniquely factors through it.

Indeed, we construct a factor $h_\infty : (X_\infty, s_\infty) \to (W, u)$ of $h_0 : (X_0, s_0) \to (W, u)$ through η as follows:



where

- for $n \geq 0$, $h_{n+1} : (X_{n+1}, s_{n+1}) \to (W, u)$ is the factor of h_n through the algebraic coequalizer e_n; and
- h_∞ is the mediating map from the colimit X_∞ to W with respect to the cone $\{ h_n \}_{n \geq 0}$.

Then, $h_\infty \circ \eta = h_0$ and, as $u \circ \Sigma h_\infty \circ \Sigma \overline{e}_n = h_\infty \circ s_\infty \circ \Sigma \overline{e}_n$ for all $n \geq 0$, it follows that h_∞ is a homomorphism $(X_\infty, s_\infty) \to (W, u)$.

We finally establish the uniqueness of such factorizations: for any homomorphism $h : (X_\infty, s_\infty) \to (W, u)$ such that $h \circ \eta = h_0$, it follows by induction that $h \circ \overline{e}_n = h_n$ for all $n \geq 0$, and hence that $h = h_\infty$. □

4.4 Inductive free $\mathbb{S}$-algebras

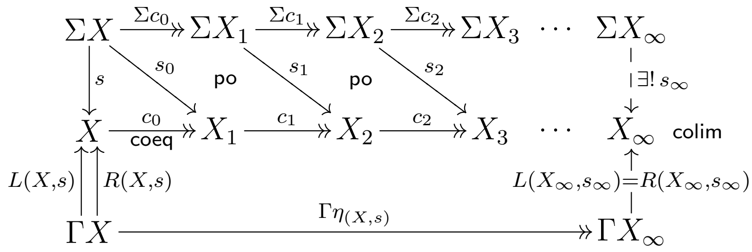
As we have seen above, free $\mathbb{S}$-algebras on Σ-algebras may be constructed by a colimit of an ω-chain of algebraic coequalizers (Theorem 4.7), each of which is in turn constructed by a coequalizer and a colimit of an ω-chain (Corollary 4.5). Here we introduce an extra condition on the functorial signature and functorial context of an equational system to accomplish the construction of free algebras in just ω steps.

Theorem 4.8 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system for which $\mathscr{C}$ is finitely cocomplete. If $\mathscr{C}$ has colimits of ω-chains and Σ preserves them, and both Σ and Γ preserve epimorphisms, then $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**.*

PROOF. Consider the construction (3) exhibiting a free $\mathbb{S}$-algebra on the Σ-algebra (X_0, s_0). According to Corollary 4.5, the algebraic coequalizer $e_0 : (X_0, s_0) \to (X_1, s_1)$ in there is an epimorphism in $\mathscr{C}$. Thus, so is Γe_0 and, as $L(X_1, s_1) \circ \Gamma e_0 = R(X_1, s_1) \circ \Gamma e_0$, it follows that (X_1, s_1) is an $\mathbb{S}$-algebra; the free one on (X_0, s_0). □

Thus, under the hypothesis of Theorem 4.8, the construction of the free $\mathbb{S}$-algebra (X_∞, s_∞) on a Σ-algebra (X, s) with unit $\eta_{(X,s)} : (X, s) \to (X_\infty, s_\infty)$

is simplified as in the following diagram:



(4)

The intuition behind the construction of X_1 from X as the coequalizer of $L(X,s)$ and $R(X,s)$ is that of quotienting the carrier object X by the equation $L = R$. The construction of X_{n+1} from X_n for $n \geq 1$ as a pushout is intuitively quotienting the object X_n by congruence rules. Therefore, the intuition behind the construction of the free algebra X_∞ is that of quotienting the object X by both the equation $L = R$ and the congruence rules.

5 Transfinite free constructions for equational systems

This technical section extends the finitary constructions and results of the previous section to the transfinite case. Overall, the following theorem is established.

Theorem 5.1 *Let* $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ *be an equational system. For* $\mathscr{C}$ *finitely and chain cocomplete, if either of the following conditions hold*

(1) Σ *and* Γ *preserve colimits of* κ*-chains for some limit ordinal* κ*;*

(2) Σ *preserves colimits of* κ*-chains for some limit ordinal* κ*, and both* Σ *and* Γ *preserve epimorphisms;*

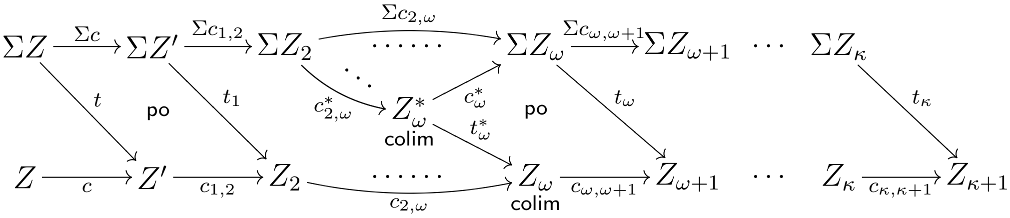
(3) $\mathscr{C}$ *has no transfinite chain of proper epimorphisms, and* Σ *preserves epimorphisms and colimits of* κ*-chains for some limit ordinal* κ

then the forgetful functor $\mathbb{S}$-**Alg** $\to \mathscr{C}$ *has a left adjoint.*

Remark *In item (3) above, we take a transfinite chain in a category* $\mathscr{C}$ *to be an* **Ord***-indexed diagram for* **Ord** *the large linear order of ordinals. Main examples of categories with no transfinite chain of proper epimorphisms are those that are well-copowered.*

Analogously to the development in Section 4, we consider the construction of algebraic coequalizers, free Σ-algebras, and free $\mathbb{S}$-algebras in turn.

Let Σ be an endofunctor on a category $\mathscr{C}$, and let $(c : Z \to Z' \leftarrow \Sigma Z : t)$ be a given Σ-algebra cospan. For κ an ordinal, we proceed to consider a (possibly transfinite) construction as depicted below



(∗)

yielding a chain $\{ c_{\alpha,\beta} : Z_\alpha \to Z_\beta \}_{\alpha\leq\beta\leq\kappa+1}$ (with $c_{0,1} = c$) and morphisms $\{ t_\alpha : \Sigma Z_\alpha \to Z_{\alpha+1} \}_{\alpha\leq\kappa}$ (with $t_0 = t$) such that

$$\begin{array}{ccc} \Sigma Z_\alpha & \xrightarrow{\Sigma c_{\alpha,\beta}} & \Sigma Z_\beta \\ {\scriptstyle t_\alpha}\downarrow & & \downarrow{\scriptstyle t_\beta} \\ Z_{\alpha+1} & \xrightarrow[c_{\alpha+1,\beta+1}]{} & Z_{\beta+1} \end{array} \tag{5}$$

commutes.

Precisely, the definitions are as follows: for $\lambda \leq \kappa$,

- when $\lambda = 0$,
  $Z_\lambda \overset{c_{\lambda,\lambda+1}}{\longrightarrow} Z_{\lambda+1} \overset{t_\lambda}{\longleftarrow} \Sigma Z_\lambda$ is $Z \overset{c}{\longrightarrow} Z' \overset{t}{\longleftarrow} \Sigma Z$;
- when λ is a successor ordinal $\alpha + 1$,
  $Z_\lambda \overset{c_{\lambda,\lambda+1}}{\longrightarrow} Z_{\lambda+1} \overset{t_\lambda}{\longleftarrow} \Sigma Z_\lambda$ is a pushout of $Z_{\alpha+1} \overset{t_\alpha}{\longleftarrow} \Sigma Z_\alpha \overset{\Sigma c_{\alpha,\alpha+1}}{\longrightarrow} \Sigma Z_{\alpha+1}$; and
- when λ is a limit ordinal,
  $Z_\lambda \overset{c_{\lambda,\lambda+1}}{\longrightarrow} Z_{\lambda+1} \overset{t_\lambda}{\longleftarrow} \Sigma Z_\lambda$ is a pushout of $Z_\lambda \overset{t^*_\lambda}{\longleftarrow} Z^*_\lambda \overset{c^*_\lambda}{\longrightarrow} \Sigma Z_\lambda$, where $\{ c_{\alpha,\lambda} : Z_\alpha \to Z_\lambda \}_{\alpha<\lambda}$ and $\{ c^*_{\alpha,\lambda} : \Sigma Z_\alpha \to Z^*_\lambda \}_{\alpha<\lambda}$ are respectively colimits of the λ-chains $\{ c_{\alpha,\beta} \}_{\alpha\leq\beta<\lambda}$ and $\{ \Sigma c_{\alpha,\beta} \}_{\alpha\leq\beta<\lambda}$, and where c^*_λ and t^*_λ are respectively the mediating maps from the colimiting cone $\{ c^*_{\alpha,\lambda} \}_{\alpha<\lambda}$ to the cones $\{ \Sigma c_{\alpha,\lambda} \}_{\alpha<\lambda}$ and $\{ c_{\alpha+1,\lambda} \circ t_\alpha \}_{\alpha<\lambda}$ of the λ-chain $\{ \Sigma c_{\alpha,\beta} \}_{\alpha\leq\beta<\lambda}$.

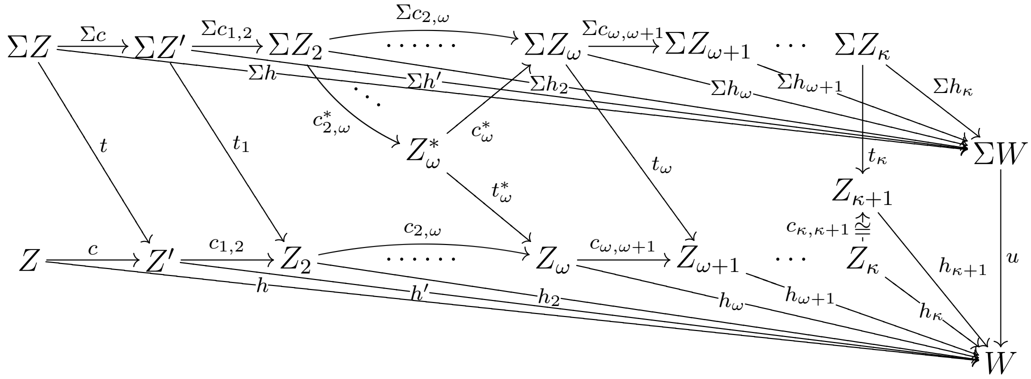
Whenever this construction can be performed for the ordinal κ, we say that it *reaches* κ. Furthermore, we say that the construction (∗) *stops* if it does so at some ordinal κ in the sense that it reaches κ and the map $c_{\kappa,\kappa+1} : Z_\kappa \to Z_{\kappa+1}$ is an isomorphism.

Theorem 5.2 *Let Σ be an endofunctor on a category $\mathscr{C}$. For a Σ-algebra cospan $(c : Z \to Z' \leftarrow \Sigma Z : t)$, if the construction (∗) for it stops, then a free*

Σ*-algebra on it exists. If, in addition, the endofunctor* Σ *preserves epimorphisms and the map* c *is epimorphic in* $\mathscr{C}$*, then the components of the universal map from the* Σ*-algebra cospan to the free* Σ*-algebra are epimorphic in* $\mathscr{C}$*.*

PROOF. Let $(Z \xrightarrow{c} Z' \xleftarrow{t} \Sigma Z)$ be a Σ-algebra cospan and assume that the construction $(*)$ for it stops at an ordinal κ. We claim that the Σ-algebra $(Z_\kappa, (c_{\kappa,\kappa+1})^{-1} \circ t_\kappa : \Sigma Z_\kappa \to Z_\kappa)$ is free over $(c : Z \to Z' \leftarrow \Sigma Z : t)$. Indeed, we show that $(c_{0,\kappa}, c_{1,\kappa}) : (Z_0 \to Z_1 \leftarrow \Sigma Z_0) \longrightarrow (Z_\kappa \xrightarrow{\mathrm{id}} Z_\kappa \leftarrow \Sigma Z_\kappa)$ is a universal map in Σ-**AlgCoSpan**.

First, note that $(c_{0,\kappa}, c_{1,\kappa})$ is indeed a map in Σ-**AlgCoSpan**; as we have that $(c_{\kappa,\kappa+1})^{-1} \circ t_\kappa \circ \Sigma c_{0,\kappa} = (c_{\kappa,\kappa+1})^{-1} \circ c_{1,\kappa+1} \circ t_0 = c_{1,\kappa} \circ t_0$. Second, consider a map $(h, h') : (Z \xrightarrow{c} Z' \xleftarrow{t} \Sigma Z) \longrightarrow (W \xrightarrow{\mathrm{id}} W \xleftarrow{u} \Sigma W)$ and perform the following (possibly transfinite) construction:



where

- for $\lambda = 0$,
  h_λ is h and $h_{\lambda+1}$ is h';
- for a successor ordinal $\lambda = \alpha + 1$,
  h_λ is $h_{\alpha+1}$, and $h_{\lambda+1}$ is the mediating map from the pushout $Z_{\lambda+1}$ to W with respect to the cone $(h_\lambda : Z_\lambda \to W \leftarrow \Sigma Z_\lambda : u \circ \Sigma h_\lambda)$ of the span $(t_\alpha : Z_{\alpha+1} \leftarrow \Sigma Z_\alpha \to \Sigma Z_{\alpha+1} : \Sigma c_{\alpha,\alpha+1})$; and
- for a limit ordinal λ,
  h_λ is the mediating map from the colimit Z_λ to W with respect to the cone $\{ h_\alpha \}_{\alpha<\lambda}$ of the λ-chain $\{ c_{\alpha,\beta} \}_{\alpha\leq\beta<\lambda}$, and $h_{\lambda+1}$ is the mediating map from the pushout $Z_{\lambda+1}$ to W with respect to the cone $(h_\lambda : Z_\lambda \to W \leftarrow \Sigma Z_\lambda : u \circ \Sigma h_\lambda)$ of the span $(t^*_\lambda : Z_\lambda \leftarrow Z^*_\lambda \to \Sigma Z_\lambda : c^*_\lambda)$.

As $h_\kappa \circ (c_{\kappa,\kappa+1})^{-1} \circ t_\kappa = h_{\kappa+1} \circ t_\kappa = u \circ \Sigma h_\kappa$, it follows that h_κ is a Σ-algebra homomorphism $(Z_\kappa, (c_{\kappa,\kappa+1})^{-1} \circ t_\kappa) \to (W, u)$. Hence, (h, h') factors as the composite $(h_\kappa, h_\kappa) \circ (c_{0,\kappa}, c_{1,\kappa})$.

We finally establish the uniqueness of such factorizations. For any homomorphism $g : (Z_\kappa, (c_{\kappa,\kappa+1})^{-1} \circ t_\kappa) \to (W, u)$ such that $g \circ c_{1,\kappa} = h'$, it follows by (a possibly transfinite) induction that $g \circ c_{\alpha,\kappa} = h_\alpha$ for all $\alpha \leq \kappa$, and hence that $g = h_\kappa$.

If Σ preserves epimorphisms and c is an epimorphism in $\mathscr{C}$, then, by (a possibly transfinite) induction, the morphisms $c_{\alpha,\beta}$ and $\Sigma c_{\alpha,\beta}$ are shown to be epimorphic in $\mathscr{C}$ for all ordinals $\alpha \leq \beta \leq \kappa$. Hence this is the case for $c_{0,\kappa}$ and $c_{1,\kappa}$. □

Corollary 5.3 *Let Σ be an endofunctor on a category $\mathscr{C}$ with coequalizers. If the construction $(*)$ stops for every Σ-algebra cospan $(c : Z \to Z_1 \leftarrow \Sigma Z : t)$ with c epimorphic in $\mathscr{C}$, then Σ-algebraic coequalizers exist. If, in addition, Σ preserves epimorphisms then Σ-algebraic coequalizers are epimorphic in $\mathscr{C}$.*

PROOF. Let $(Z, t : \Sigma Z \to Z)$ be a Σ-algebra and let l, r be a parallel pair into Z in $\mathscr{C}$. Consider a coequalizer $c : Z \twoheadrightarrow Z_1$ of l, r in $\mathscr{C}$ and the Σ-algebra cospan $(Z \xrightarrow{c} Z_1 \xleftarrow{ct} \Sigma Z)$ as in the proof of Lemma 4.3. As c is an epimorphism, by Theorem 5.2, a free Σ-algebra (Z', t') on $(Z \xrightarrow{c} Z_1 \xleftarrow{ct} \Sigma Z)$ exists. Let $(z, z_1) : (Z \xrightarrow{c} Z_1 \xleftarrow{ct} \Sigma Z) \longrightarrow (Z' \xrightarrow{\mathrm{id}} Z' \xleftarrow{t'} \Sigma Z')$ be the universal map. Then, the homomorphism $z = z_1 \circ c : (Z, t) \to (Z', t')$ is an algebraic coequalizer of l, r.

If Σ preserves epimorphisms, then, by Theorem 5.2, the maps z, z_1 are epimorphic in $\mathscr{C}$ as so is c. □

5.2 Free Σ-algebras

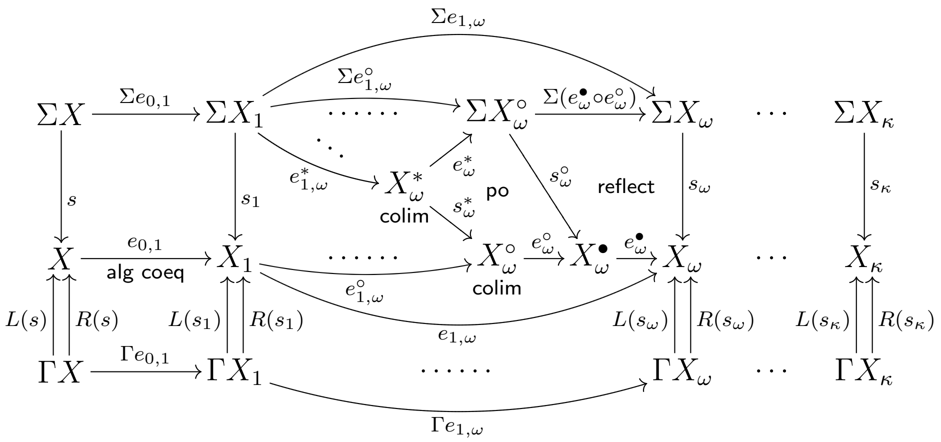
The following well-known result (see *e.g.* [2]) follows from Theorem 5.2.

Corollary 5.4 *For an endofunctor Σ on a category $\mathscr{C}$ with finite coproducts, let Σ_X, for $X \in \mathscr{C}$, be the endofunctor $X + \Sigma(-)$ on $\mathscr{C}$. For an object $X \in \mathscr{C}$, if the construction $(*)$ with respect to the endofunctor Σ_X for the initial Σ_X-algebra cospan $(0 \xrightarrow{!} \Sigma_X 0 \xleftarrow{\mathrm{id}} \Sigma_X 0)$ stops, then it yields an initial Σ_X-algebra whose Σ-algebra component is a free Σ-algebra on X.*

Note that in the above particular case of the construction $(*)$, we have that $X_0 = 0$; that $X_{\alpha+1} = X + \Sigma X_\alpha$ for all successor ordinals $\alpha + 1$; and that X_λ is a colimit of the λ-chain $\{ c_{\alpha,\beta} \}_{\alpha \leq \beta < \lambda}$ for all limit ordinals λ.

5.3 Free $\mathbb{S}$-algebras

Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system, and let (X, s) be a given Σ-algebra. For κ an ordinal, we proceed to consider a (possibly transfinite) construction as depicted below



$(**)$

yielding a chain $\{ e_{\alpha,\beta} : (X_\alpha, s_\alpha) \to (X_\beta, s_\beta) \}_{\alpha \le \beta \le \kappa}$ (with $s_0 = s$) in Σ-**Alg**.

Precisely, the definitions are as follows: for $\lambda \le \kappa$,

- when $\lambda = 0$,
  (X_λ, s_λ) is (X, s);

- when λ is a successor ordinal $\alpha + 1$,
  $e_{\alpha,\lambda} : (X_\alpha, s_\alpha) \to (X_\lambda, s_\lambda)$ is an algebraic coequalizer of the parallel pair $L(X_\alpha, s_\alpha), R(X_\alpha, s_\alpha) : \Gamma X_\alpha \to X_\alpha$; and

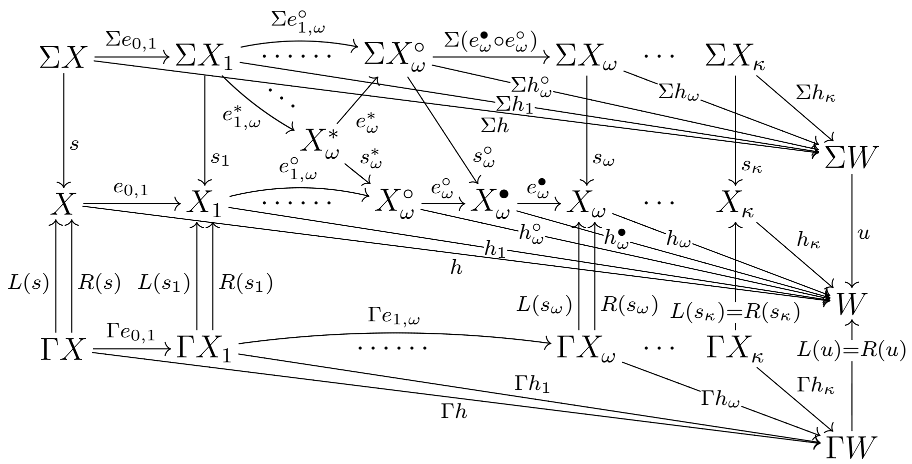
- when λ is a limit ordinal,
  - $\{ e^\circ_{\alpha,\lambda} : X_\alpha \to X^\circ_\lambda \}_{\alpha<\lambda}$ and $\{ e^*_{\alpha,\lambda} : \Sigma X_\alpha \to X^*_\lambda \}_{\alpha<\lambda}$ are respectively colimits of the λ-chains $\{ e_{\alpha,\beta} \}_{\alpha \le \beta < \lambda}$ and $\{ \Sigma e_{\alpha,\beta} \}_{\alpha \le \beta < \lambda}$;
  - $e^*_\lambda : X^*_\lambda \to \Sigma X^\circ_\lambda$ and $s^*_\lambda : X^*_\lambda \to X^\circ_\lambda$ are the mediating maps from the colimiting cone $\{ e^*_{\alpha,\lambda} \}_{\alpha<\lambda}$ to the cones $\{ \Sigma e^\circ_{\alpha,\lambda} \}_{\alpha<\lambda}$ and $\{ e^\circ_{\alpha,\lambda} \circ s_\alpha \}_{\alpha<\lambda}$;
  - $(X^\circ_\lambda \overset{e^\circ_\lambda}{\to} X^\bullet_\lambda \overset{s^\circ_\lambda}{\leftarrow} \Sigma X^\circ_\lambda)$ is a pushout of $(X^\circ_\lambda \overset{s^*_\lambda}{\leftarrow} X^*_\lambda \overset{e^*_\lambda}{\to} \Sigma X^\circ_\lambda)$;
  - (X_λ, s_λ) is a free Σ-algebra on the Σ-algebra cospan $(X^\circ_\lambda \overset{e^\circ_\lambda}{\to} X^\bullet_\lambda \overset{s^\circ_\lambda}{\leftarrow} \Sigma X^\circ_\lambda)$ with universal map $(e^\bullet_\lambda \circ e^\circ_\lambda, e^\bullet_\lambda)$; and
  - $e_{\alpha,\lambda} : X_\alpha \to X_\lambda$ is the composite $e^\bullet_\lambda \circ e^\circ_\lambda \circ e^\circ_{\alpha,\lambda}$.

Whenever this construction can be performed for the ordinal κ, we say that it *reaches* κ. Furthermore, we say that the construction $(**)$ *stops* if it does so at some ordinal κ in the sense that it reaches $\kappa + 1$ and the map $e_{\kappa,\kappa+1} : X_\kappa \to X_{\kappa+1}$ is an isomorphism.

Theorem 5.5 *Let* $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ *be an equational system. If the construction* $(**)$ *stops for every* Σ*-algebra, then* $\mathbb{S}$-**Alg** *is a full reflective subcategory of* Σ-**Alg**.

PROOF. Let (X, s) be a Σ-algebra and assume that the construction $(**)$ for it stops at an ordinal κ. We claim that the Σ-algebra (X_κ, s_κ) is a free $\mathbb{S}$-algebra on (X, s). First, note that (X_κ, s_κ) is an $\mathbb{S}$-algebra since $e_{\kappa,\kappa+1} \circ L(X_\kappa, s_\kappa) = e_{\kappa,\kappa+1} \circ R(X_\kappa, s_\kappa)$ and $e_{\kappa,\kappa+1}$ is an isomorphism. We will now show that the homomorphism $e_{0,\kappa} : (X, s) \to (X_\kappa, s_\kappa)$ is universal.

Consider a homomorphism $h : (X, s) \to (W, u)$ and perform the following (possibly transfinite) construction:



where

- for a successor ordinal $\lambda = \alpha + 1$,
  $h_\lambda : (X_\lambda, s_\lambda) \to (W, u)$ is the factor of h_α through the algebraic coequalizer $e_{\alpha,\alpha+1}$; and
- for a limit ordinal λ,
  - h°_λ is the mediating map from the colimit X°_λ to W with respect to the cone $\{ h_\alpha \}_{\alpha<\lambda}$;
  - $h^\bullet_\lambda$ is the mediating map from the pushout $X^\bullet_\lambda$ to W with respect to the cone $(h^\circ_\lambda : X^\circ_\lambda \to W \leftarrow \Sigma X^\circ_\lambda : u \circ \Sigma h^\circ_\lambda)$; and
  - $h_\lambda : (X_\lambda, s_\lambda) \to (W, u)$ is the factor of

$$(h^\circ_\lambda, h^\bullet_\lambda) : (X^\circ_\lambda \xrightarrow{e^\circ_\lambda} X^\bullet_\lambda \xleftarrow{s^\circ_\lambda} \Sigma X^\circ_\lambda) \to (W \xrightarrow{\mathrm{id}} W \xleftarrow{u} \Sigma W)$$

    through the universal map $(e^\bullet_\lambda \circ e^\circ_\lambda, e^\bullet_\lambda)$.

Thus, $h_\kappa : (X_\kappa, s_\kappa) \to (W, u)$ is a factor of $h : (X, s) \to (W, u)$ through $e_{0,\kappa} : (X, s) \to (X_\kappa, s_\kappa)$.

We finally establish the uniqueness of such factorizations. Indeed, for any homomorphism $g : (X_\kappa, s_\kappa) \to (W, u)$ such that $g \circ e_{0,\kappa} = h$, it follows by (a possibly transfinite) induction that $g \circ e_{\alpha,\kappa} = h_\alpha$ for all $\alpha \le \kappa$, and hence that $g = h_\kappa$. $\square$

We conclude the section by giving three sufficient conditions, respectively corresponding to the three conditions of Theorem 5.1, that permit the application of Corollary 5.4 and Theorem 5.5, and thus lead to transfinite constructions of free algebras for equational systems.

Theorem 5.6 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system with $\mathscr{C}$ finitely and chain cocomplete.*

If Σ preserves colimits of κ-chains for some limit ordinal κ, then the construction $(*)$ stops at κ for all Σ-algebra cospans.

In addition, if Γ preserves colimits of κ-chains, or if both Σ and Γ preserve epimorphisms, then the construction $(**)$ respectively stops at κ, or at 1, for every Σ-algebra.

PROOF. Assume that Σ preserves colimits of κ-chains for some limit ordinal κ. As $\mathscr{C}$ is finitely and chain cocomplete, the construction $(*)$ for a Σ-algebra cospan $(c : Z \to Z' \leftarrow \Sigma Z : t)$ reaches the ordinal κ. As Σ preserves the colimiting cone $\{ c_{\alpha,\kappa} \}_{\alpha<\kappa}$ of the κ-chain $\{ c_{\alpha,\beta} \}_{\alpha\leq\beta<\kappa}$, the mediating map c^*_κ is an isomorphism and hence so is $c_{\kappa,\kappa+1}$.

By Theorem 5.2, free Σ-algebras on Σ-algebraic cospans exist; and so do Σ-algebraic coequalizers by Corollary 5.3. Thus, the construction $(**)$ reaches any ordinal.

In addition, assume that Γ preserves colimits of κ-chains, and consider the construction $(**)$ for a Σ-algebra (X, s) up to the ordinal $\kappa+1$. As Σ preserves the colimiting cone $\{ e^\circ_{\alpha,\kappa} \}_{\alpha<\kappa}$ of the κ-chain $\{ e_{\alpha,\beta} \}_{\alpha\leq\beta<\kappa}$, the mediating map e^*_κ is an isomorphism and hence so are e°_κ and $e^\bullet_\kappa$. From this, we have that $\{ e_{\alpha,\kappa} \}_{\alpha<\kappa}$ is a colimiting cone of the κ-chain $\{ e_{\alpha,\beta} \}_{\alpha\leq\beta<\kappa}$. Since Γ preserves it and $L(X_\kappa, s_\kappa) \circ \Gamma e_{\alpha,\kappa} = R(X_\kappa, s_\kappa) \circ \Gamma e_{\alpha,\kappa}$ for all $\alpha < \kappa$, it follows that $L(X_\kappa, s_\kappa) = R(X_\kappa, s_\kappa)$. Consequently, the algebraic coequalizer $e_{\kappa,\kappa+1}$ is an isomorphism.

Alternatively, besides Σ preserving colimits of κ-chains, assume both that Σ and Γ preserve epimorphisms, and consider the construction $(**)$ for a Σ-algebra (X, s) up to the ordinal 2. Then, by Corollary 5.3, the Σ-algebraic coequalizer $e_{0,1}$ is epimorphic in $\mathscr{C}$, and thus so is $\Gamma e_{0,1}$. Moreover, since $\Gamma e_{0,1}$ equalizes $L(X_1, s_1), R(X_1, s_1)$, it follows that $L(X_1, s_1) = R(X_1, s_1)$. As a result, the algebraic coequalizer $e_{1,2}$ is an isomorphism. □

Theorem 5.7 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system with $\mathscr{C}$ finitely and chain cocomplete. If $\mathscr{C}$ has no transfinite chain of proper epimorphisms and Σ preserves epimorphisms, then the construction $(*)$ stops* *for all Σ-algebra cospans $(c : Z \to Z' \leftarrow \Sigma Z : t)$ with c epimorphic, and the construction $(**)$ stops for all Σ-algebras.*

PROOF. As $\mathscr{C}$ is finitely and chain cocomplete, the construction $(*)$ for a Σ-algebra cospan $(c : Z \to Z' \leftarrow \Sigma Z : t)$ with c epimorphic reaches every ordinal. As Σ preserves epimorphisms, it follows by transfinite induction that the maps $c_{\alpha,\beta}$ and $\Sigma c_{\alpha,\beta}$ are epimorphic for all ordinals $\alpha \leq \beta$. Since $\{ c_{\alpha,\beta} \}_{\alpha \leq \beta \in \mathbf{Ord}}$ is a transfinite chain of epimorphisms, there exists, by hypothesis, an isomorphic component $c_{\alpha,\beta}$ for some pair of ordinals $\alpha < \beta$. Thus the construction stops.

By Theorem 5.2 and Corollary 5.3, free Σ-algebras on Σ-algebraic cospans $(c : Z \to Z' \leftarrow \Sigma Z : t)$ with c epimorphic and Σ-algebraic coequalizers exist, and their associated universal maps are epimorphic in $\mathscr{C}$. Consequently, it follows that the construction $(**)$ reaches every ordinal for all Σ-algebras and, by transfinite induction, that the maps $e_{\alpha,\beta}$, $e^{\circ}_{\kappa,\lambda}$, $e^{*}_{\kappa,\lambda}$, e^{*}_{λ}, e°_{λ}, $e^{\bullet}_{\lambda}$ are epimorphisms in $\mathscr{C}$, for all $\alpha \leq \beta \in \mathbf{Ord}$ and $\kappa < \lambda \in \mathbf{Ord}$ with λ a limit ordinal. Since $\{ e_{\alpha,\beta} \}_{\alpha \leq \beta \in \mathbf{Ord}}$ is a transfinite chain of epimorphisms, there exists, by hypothesis, an isomorphic component $e_{\alpha,\beta}$ for some pair of ordinals $\alpha < \beta$. Thus the construction stops. □

The following two corollaries imply Theorem 5.1.

Corollary 5.8 *Let Σ be an endofunctor on a category $\mathscr{C}$. For $\mathscr{C}$ finitely and chain cocomplete, if Σ preserves colimits of κ-chains for some limit ordinal κ, then Σ-**Alg** is a full reflective subcategory of Σ-**AlgCoSpan** and the forgetful functor Σ-**Alg** $\to \mathscr{C}$ has a left adjoint.*

PROOF. The first conclusion follows from Theorems 5.6 and 5.2; the second one from Theorem 5.6 and Corollary 5.4 (as the endofunctors $X + \Sigma(-)$ preserve colimits of κ-chains for all $X \in \mathscr{C}$). □

Corollary 5.9 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ finitely and chain cocomplete, if either of the following conditions hold*

(1) Σ and Γ preserve colimits of κ-chains for some limit ordinal κ;
(2) Σ preserves colimits of κ-chains for some limit ordinal κ, and both Σ and Γ preserve epimorphisms;
(3) $\mathscr{C}$ has no transfinite chain of proper epimorphisms and Σ preserves epimorphisms

*then $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**.*

PROOF. Items (1) and (2) follow from Theorems 5.6 and 5.5; item (3) follows from Theorems 5.7 and 5.5. □

6 Categories of algebras and monads for equational systems

We consider properties of categories of algebras and monads for equational systems. The preceding results and those of this section jointly establish the following theorems.

Theorem 6.1 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if Σ and Γ preserve colimits of κ-chains for some limit ordinal κ then the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic, the induced monad preserves colimits of κ-chains, and $\mathbb{S}\text{-}\mathbf{Alg}$ is cocomplete.*

Theorem 6.2 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if Σ preserves both epimorphisms and colimits of κ-chains for some limit ordinal κ, and if Γ preserves either epimorphisms or colimits of κ-chains, then the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic, the induced monad preserves epimorphisms, and $\mathbb{S}\text{-}\mathbf{Alg}$ is cocomplete.*

Theorem 6.3 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system such that the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint. If $\mathscr{C}$ is cocomplete and has no transfinite chain of proper epimorphisms, and Σ preserves epimorphisms, then $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic and $\mathbb{S}\text{-}\mathbf{Alg}$ is cocomplete.*

Remark *Theorem 6.1 and Theorem 6.2 follow from Corollaries 6.6 and 6.8, and Proposition 6.10; Theorem 6.3 follows from Propositions 6.4 and 6.5, Corollary 5.9, Theorem 5.7, and Corollary 5.3.*

6.1 Monadicity and cocompleteness

For an endofunctor Σ on a category $\mathscr{C}$, it is well known that if the forgetful functor $\Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint then it is monadic. This result extends to categories of algebras for equational systems.

Proposition 6.4 *Let $\mathbb{S}$ be an equational system. If the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, then it is monadic.*

PROOF. To show the monadicity of $U_{\mathbb{S}}$ by Beck's theorem [25], it is enough to show that $U_{\mathbb{S}}$ creates coequalizers of parallel pairs $f, g : (X, r) \to (Y, s)$ in $\mathbb{S}\text{-}\mathbf{Alg}$ for which $f, g : X \to Y$ has an absolute coequalizer, say $e : Y \twoheadrightarrow Z$, in $\mathscr{C}$. In this case, then, Σe is a coequalizer of $\Sigma f, \Sigma g$ and Γe is a coequalizer of $\Gamma f, \Gamma g$, so that we have the following situation

$$\begin{array}{ccccc}
\Sigma X & \overset{\Sigma f}{\underset{\Sigma g}{\rightrightarrows}} & \Sigma Y & \overset{\Sigma e}{\twoheadrightarrow} & \Sigma Z \\
{\scriptstyle r}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \exists!\, t} \\
X & \overset{f}{\underset{g}{\rightrightarrows}} & Y & \overset{e}{\underset{\mathsf{coeq}}{\twoheadrightarrow}} & Z \\
\uparrow{\scriptstyle L(X,r)=R(X,r)} & & \uparrow{\scriptstyle L(Y,s)=R(Y,s)} & & {\scriptstyle L(Z,t)}\uparrow\uparrow{\scriptstyle R(Z,t)} \\
\Gamma X & \overset{\Gamma f}{\underset{\Gamma g}{\rightrightarrows}} & \Gamma Y & \overset{\Gamma e}{\twoheadrightarrow} & \Gamma Z
\end{array}$$

for a unique Σ-algebra structure t on Z for which $L(Z,t) = R(Z,t)$.

It follows from the universal properties of e and Σe that $e : (Y,s) \to (Z,t)$ is a coequalizer of $f, g : (X,r) \to (Y,s)$ in Σ-**Alg**, and hence also in $\mathbb{S}$-**Alg**. □

A general condition for the cocompleteness of categories of algebras for equational systems follows.

Proposition 6.5 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system with $\mathscr{C}$ cocomplete. If the forgetful functor $U_{\mathbb{S}} : \mathbb{S}$-**Alg** $\to \mathscr{C}$ has a left adjoint, $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**, and Σ-**Alg** has coequalizers, then the category $\mathbb{S}$-**Alg** is cocomplete.*

PROOF. $\mathbb{S}$-**Alg** has coequalizers since it is a full reflective subcategory of Σ-**Alg**, which is assumed to have coequalizers. Also, by Proposition 6.4, $\mathbb{S}$-**Alg** is monadic over $\mathscr{C}$. Being monadic over a cocomplete category and having coequalizers, $\mathbb{S}$-**Alg** is cocomplete (see *e.g.* [5, Proposition 4.3.4]). □

Since the existence of Σ-algebraic coequalizers implies that of coequalizers in Σ-**Alg**, we obtain the following corollary.

Corollary 6.6 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if either of the following conditions hold*

(1) Σ and Γ preserve colimits of κ-chains for some infinite limit ordinal κ;
(2) Σ preserves colimits of κ-chains for some limit ordinal κ, and both Σ and Γ preserve epimorphisms;
(3) $\mathscr{C}$ has no transfinite chain of proper epimorphisms, and Σ preserves epimorphisms and colimits of κ-chains for some infinite limit ordinal κ

*then the forgetful functor $U_{\mathbb{S}} : \mathbb{S}$-**Alg** $\to \mathscr{C}$ is monadic and $\mathbb{S}$-**Alg** is cocomplete.*

We show that the colimit-preservation properties of the functorial signature and functorial context of an equational system are inherited by the free-algebra monad.

Recall that a diagram in a category $\mathscr{C}$ is a functor from a small category to $\mathscr{C}$. We say that a class $\mathcal{D}$ of diagrams in $\mathscr{C}$ is closed under an endofunctor F on $\mathscr{C}$ if the diagram $F \circ I : \mathbb{I} \to \mathscr{C}$ is in $\mathcal{D}$ for all diagrams $I : \mathbb{I} \to \mathscr{C}$ in $\mathcal{D}$.

Proposition 6.7 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system for which $U_{\mathbb{S}} : \mathbb{S}$-**Alg** $\to \mathscr{C}$ has a left adjoint, and write (T, η, μ) for the induced monad on $\mathscr{C}$. For $\mathcal{D}$ a class of diagrams in $\mathscr{C}$ closed under T, if $\mathscr{C}$ has colimits of diagrams in $\mathcal{D}$ and the endofunctors Σ and Γ preserve them, then so does the endofunctor T.*

PROOF. For a diagram $I : \mathbb{I} \to \mathscr{C}$ in $\mathcal{D}$, let $\{\, \lambda_i : Ii \to \operatorname{colim} I \,\}_{i \in \mathbb{I}}$ and $\{\, \delta_i : TIi \to \operatorname{colim} TI \,\}_{i \in \mathbb{I}}$ be colimiting cones. We show that the cones $\{\, T\lambda_i \,\}_{i \in \mathbb{I}}$ and $\{\, \delta_i \,\}_{i \in \mathbb{I}}$ are isomorphic. Specifically, we construct an inverse $q : T(\operatorname{colim} I) \to \operatorname{colim} TI$ to the mediating map $p : \operatorname{colim} TI \to T(\operatorname{colim} I)$ from $\{\, \delta_i \,\}_{i \in \mathbb{I}}$ to $\{\, T\lambda_i \,\}_{i \in \mathbb{I}}$ as follows.

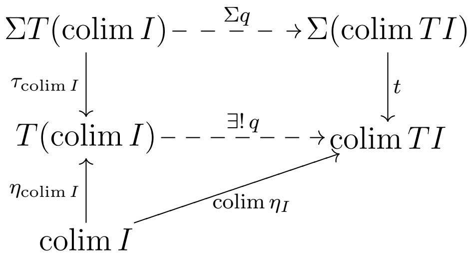
Let $(TX, \tau_X : \Sigma TX \to TX)$ be the free $\mathbb{S}$-algebra on $X \in \mathscr{C}$ induced by the left adjoint to $U_{\mathbb{S}}$. The family $\tau = \{\, \tau_X : \Sigma TX \to TX \,\}_{X \in \mathscr{C}}$ is natural. Hence, the family $\{\, \delta_i \circ \tau_{Ii} : \Sigma TIi \to \operatorname{colim} TI \,\}_{i \in \mathbb{I}}$ is a cone and, as $\{\, \Sigma\delta_i \,\}_{i \in \mathbb{I}}$ is colimiting, we have a unique Σ-algebra structure t on $\operatorname{colim} TI$ such that the diagram on the top below

$$\begin{array}{ccc} \Sigma TIi & \xrightarrow{\Sigma\delta_i} & \Sigma(\operatorname{colim} TI) \\ \downarrow{\scriptstyle \tau_{Ii}} & & \downarrow{\scriptstyle \exists!\, t} \\ TIi & \xrightarrow{\delta_i} & \operatorname{colim} TI \\ \uparrow{\scriptstyle L(TIi,\tau_{Ii})=R(TIi,\tau_{Ii})} & & \uparrow\uparrow{\scriptstyle L(\operatorname{colim} TI,t)\,|\,R(\operatorname{colim} TI,t)} \\ \Gamma TIi & \xrightarrow{\Gamma\delta_i} & \Gamma(\operatorname{colim} TI) \end{array}$$

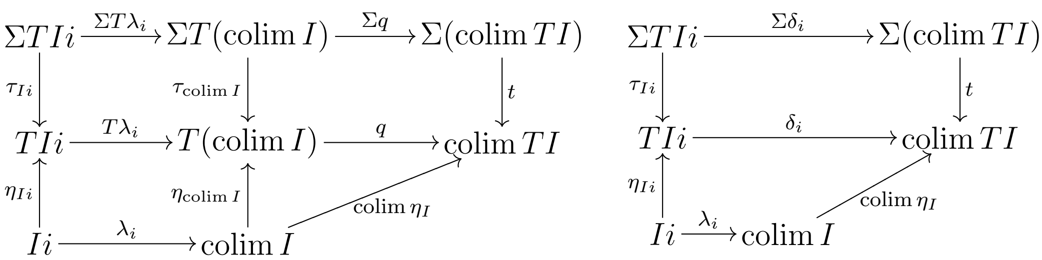
commutes for all $i \in \mathbb{I}$. Furthermore, the Σ-algebra $(\operatorname{colim} TI, t)$ is an $\mathbb{S}$-algebra; since $\{\, \Gamma\delta_i \,\}_{i \in \mathbb{I}}$ is colimiting and $L(\operatorname{colim} TI, t) \circ \Gamma\delta_i = R(\operatorname{colim} TI, t) \circ \Gamma\delta_i$ for all $i \in \mathbb{I}$.

By the universal property of free algebras, we define $q : T(\operatorname{colim} I) \to \operatorname{colim} TI$

as the unique map making the following diagram commutative:

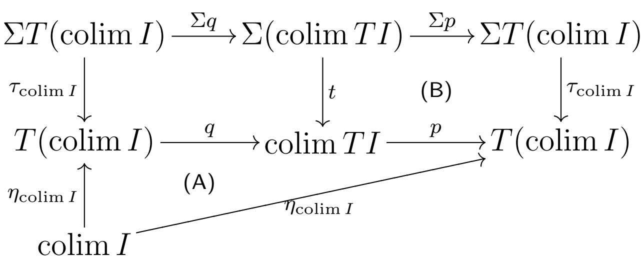


This map is a morphism between the cones $\{ T\lambda_i \}_{i\in\mathbb{I}}$ and $\{ \delta_i \}_{i\in\mathbb{I}}$; as follows from the commutative diagrams below



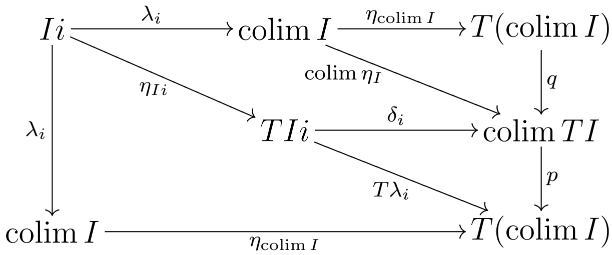
by the universal property of free algebras. It further follows that the endomap $q \circ p$ on $(\operatorname{colim} TI)$ is the identity, as it is an endomap on a colimiting cone.

Finally, that the endomap $p \circ q$ on $T(\operatorname{colim} I)$ is the identity follows from the commutativity of the diagram below



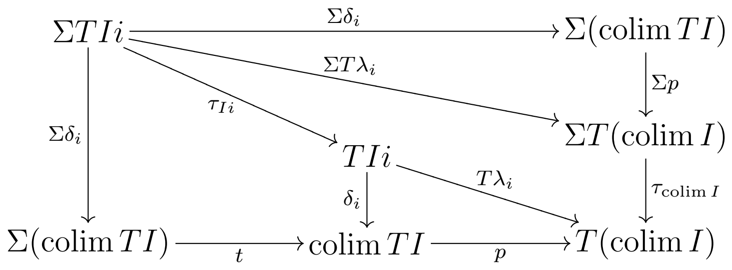
by the universal property of free algebras.

The commutativity of the diagram (A) above follows from the commutativity of the following diagram for each $i \in \mathbb{I}$



because $\{ \lambda_i \}_{i \in \mathbb{I}}$ is a colimiting cone.

The commutativity of diagram (B) above follows from the commutativity of the following diagram for each $i \in \mathbb{I}$



because $\{ \Sigma\delta_i \}_{i \in \mathbb{I}}$ is a colimiting cone. □

Corollary 6.8 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ finitely and chain cocomplete, if the endofunctors Σ and Γ preserve colimits of κ-chains for some limit ordinal κ then so does the monad induced by the left adjoint to the forgetful functor $U_\mathbb{S} : \mathbb{S}$-**Alg** $\to \mathscr{C}$.*

6.3 Epicontinuity

Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system for which the forgetful functor $U_\mathbb{S} : \mathbb{S}$-**Alg** $\to \mathscr{C}$ has a left adjoint, and write (T, η, μ) for the induced monad on $\mathscr{C}$. It follows from Proposition 6.7 that if Σ and Γ preserve cokernel pairs (*viz.*, pushouts of spans with identical legs) then so does T; so that, in particular, it also preserves epimorphisms. However, under the free constructions of Sections 4 and 5, one can directly obtain epicontinuity.

Proposition 6.9 *Let Σ be an endofunctor on a category $\mathscr{C}$, and assume that $\mathscr{C}$ is finitely and chain cocomplete and that Σ preserves colimits of κ-chains for some limit ordinal κ. If Σ preserves epimorphisms then so does the monad T_Σ induced by the left adjoint to the forgetful functor $U_\Sigma : \Sigma$-**Alg** $\to \mathscr{C}$.*

PROOF. Recall from Theorem 5.6 and Corollary 5.4 that the construction that sets

- for $\lambda = 0$,
  $(X_\lambda \to X_{\lambda+1}) = (0 \to X + \Sigma 0)$;
- for $\lambda = \alpha + 1$ a successor ordinal,
  $(X_\lambda \to X_{\lambda+1}) = (X + \Sigma X_\alpha \to X + \Sigma X_\lambda)$; and

- for λ a limit ordinal,
  $X_\lambda \to X_{\lambda+1}$ to be the mediating map from a colimiting cone $\{ X_{\alpha+1} \to X_\lambda \}_{\alpha<\lambda}$ to the cone $\{ X_{\alpha+1} = X + \Sigma X_\alpha \to X + \Sigma X_\lambda = X_{\lambda+1} \}$

stops at κ, in that the map $X_\kappa \to X_{\kappa+1}$ is an isomorphism, and yields an initial $(X + \Sigma(-))$-algebra $(X_\kappa, [\eta_X, \tau_X] : X + \Sigma X_\kappa \xrightarrow{\cong} X_\kappa)$ whose component $\tau_X : \Sigma X_\kappa \to X_\kappa$ is a free Σ-algebra on X.

Given an epimorphism $f : X \twoheadrightarrow Y$, one constructs a family of epimorphisms $\{ f_\alpha : X_\alpha \twoheadrightarrow Y_\alpha \}_{\alpha \le \kappa}$ such that

$$\begin{array}{ccc} X_\alpha & \longrightarrow & X_\beta \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_\beta} \\ Y_\alpha & \longrightarrow & Y_\beta \end{array}$$

by setting $f_0 = \mathrm{id}$; $f_{\alpha+1} = f + \Sigma f_\alpha$, for all successor ordinals $\alpha + 1$; and f_λ to be the unique mediating map from the colimiting cone $\{ X_\alpha \to X_\lambda \}_{\alpha<\lambda}$ to the cone $\{ X_\alpha \xrightarrow{f_\alpha} Y_\alpha \to Y_\lambda \}_{\alpha<\lambda}$, for all limit ordinals λ.

$$\begin{array}{ccccccccccccc} 0 & \xrightarrow{!} & X + \Sigma 0 & \xrightarrow{X+\Sigma !} & X + \Sigma(X + \Sigma 0) & \cdots & X_\omega & \longrightarrow & X + \Sigma X_\omega & \cdots & X_\kappa = T_\Sigma X \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f+\Sigma 0} & & \downarrow{\scriptstyle f+\Sigma(f+\Sigma 0)} & & \downarrow{\scriptstyle f_\omega} & & \downarrow{\scriptstyle f+\Sigma f_\omega} & & \downarrow{\scriptstyle f_\kappa = T_\Sigma f} \\ 0 & \xrightarrow{!} & Y + \Sigma 0 & \xrightarrow{Y+\Sigma !} & Y + \Sigma(Y + \Sigma 0) & \cdots & Y_\omega & \longrightarrow & Y + \Sigma Y_\omega & \cdots & Y_\kappa = T_\Sigma Y \\ & & & & & & \text{colim} & & & & \end{array}$$

By construction, and because the maps $X_\kappa \to X_{\kappa+1}$ and $Y_\kappa \to Y_{\kappa+1}$ are isomorphisms, it follows that $f_\kappa \circ \eta_X = \eta_Y \circ f$ and $f_\kappa \circ \tau_X = \tau_Y \circ \Sigma f_\kappa$. Thus, $T_\Sigma f = f_\kappa$ is an epimorphism. □

Proposition 6.10 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system for which $\mathscr{C}$ is finitely and chain cocomplete. If Σ preserves both epimorphisms and colimits of κ-chains for some limit ordinal κ, and if Γ preserves either epimorphisms or colimits of κ-chains, then the monad $T_{\mathbb{S}}$ induced by the left adjoint to the forgetful functor $U_{\mathbb{S}} : \mathbb{S}$-**Alg** $\to \mathscr{C}$ preserves epimorphisms.*

PROOF. For $X \in \mathscr{C}$, the free $\mathbb{S}$-algebra $(T_{\mathbb{S}}X, \widetilde{\tau}_X)$ over the free Σ-algebra $(T_\Sigma X, \tau_X)$ on X is given by means of the constructions $(*)$ and $(**)$; and, as Σ preserves epimorphisms, it follows that the universal homomorphism $q_X : (T_\Sigma X, \tau_X) \to (T_{\mathbb{S}}X, \widetilde{\tau}_X)$ is epimorphic in $\mathscr{C}$. Then, using Proposition 6.9, for every epimorphism $f : X \twoheadrightarrow Y$, we have the following situation

$$\begin{array}{ccc} T_\Sigma X & \overset{q_X}{\twoheadrightarrow} & T_{\mathbb{S}}X \\ {\scriptstyle T_\Sigma f}\downarrow & \searrow & \downarrow{\scriptstyle T_{\mathbb{S}} f} \\ T_\Sigma Y & \underset{q_Y}{\twoheadrightarrow} & T_{\mathbb{S}}Y \end{array}$$

and so $T_{\mathbb{S}}f$ is an epimorphism. □

We revisit the examples of equational systems given in Section 3 in the light of the above results.

(1) For the equational system $\mathbb{S}_{\mathbb{T}} = (\mathbf{Set} : \Sigma_{\mathbb{T}} \rhd \Gamma_{\mathbb{T}} \vdash L_{\mathbb{T}} = R_{\mathbb{T}})$ representing an algebraic theory $\mathbb{T}$, the category $\mathbb{S}_{\mathbb{T}}$-**Alg** is monadic over **Set** and cocomplete, and the free-algebra monad is finitary (*i.e.*, preserves filtered colimits, or equivalently, colimits of λ-chains for all limit ordinals λ) by Theorem 6.1, as **Set** is cocomplete and $\Sigma_{\mathbb{T}}$ and $\Gamma_{\mathbb{T}}$ are finitary. Furthermore, $\Sigma_{\mathbb{T}}$ and $\Gamma_{\mathbb{T}}$ preserve epimorphisms, and hence Theorems 4.8 and 6.2 apply.

(2) For the equational system $\mathbb{S}_{\mathbb{T}} = (\mathscr{C}_0 : (GB)_0 \rhd (GE)_0 \vdash \overline{\sigma}_0 = \overline{\tau}_0)$ representing an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, B, E, \sigma, \tau)$, the category $\mathbb{S}_{\mathbb{T}}$-**Alg** is monadic over $\mathscr{C}_0$ and cocomplete, and the free-algebra monad is finitary by Theorem 6.1, as $\mathscr{C}_0$ is locally finitely presentable and thus cocomplete, and $(GB)_0$ and $(GE)_0$ are finitary.

(3) One may apply Theorem 6.1 to the equational system $\mathbb{S}_{\mathbf{T}}$ representing a monad $\mathbf{T} = (T, \eta, \mu)$ on a cocomplete category $\mathscr{C}$ as follows. If T preserves colimits of λ-chains for some limit ordinal λ, then $\mathbb{S}_{\mathbf{T}}$-**Alg** $\cong \mathscr{C}^{\mathbf{T}}$ is cocomplete.

    One may also apply Theorem 6.3 as follows. If $\mathscr{C}$ has no transfinite chain of proper epimorphisms and T preserves epimorphisms, then $\mathbb{S}_{\mathbf{T}}$-**Alg** $\cong \mathscr{C}^{\mathbf{T}}$ is cocomplete.

(4) To the equational system $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$ of monoids in a monoidal cocomplete category $\mathscr{C}$ we can apply Theorem 6.1 as follows. If the tensor product is finitary (as it happens, for instance, when it is biclosed) then $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$-**Alg** is monadic over $\mathscr{C}$ and cocomplete, and the free-monoid monad is finitary. If the tensor product also preserves epimorphisms (again, as it happens when it is biclosed) then so does the free-monoid monad, by Theorem 6.2.

7 Applications

This section illustrates the theory of equational systems with three sample modern applications: (*i*) pi-calculus algebras (Section 7.1); (*ii*) binding algebras with substitution structure (Section 7.2); and (*iii*) nominal equational theories (Section 7.3).

Our presentation discusses the difficulties in representing these mathematical structures as enriched algebraic theories, and shows how these are overcome by equational systems. The theory of equational systems is then applied to study the applications.

7.1 Pi-calculus algebras

π-algebras are an algebraic model of the finitary π-calculus introduced by Stark in [32]. Here we briefly discuss the concept as algebras for an equational system. The theory of equational systems is then applied to deduce the existence of free models.

We need consider the presheaf category $\mathbf{Set}^{\mathbb{I}}$, for $\mathbb{I}$ the (essentially small) category of finite sets and injections. The category $\mathbf{Set}^{\mathbb{I}}$ carries an affine doubly closed structure (see [30]) given by:

- the cartesian closed structure $\big(1, \times, (=)^{(-)}\big)$, and
- the symmetric monoidal closed structure $\big(1, \otimes, (-) \multimap (=)\big)$ induced by Day's construction [9] from the symmetric monoidal structure $(\emptyset, \uplus)$ on $\mathbb{I}^{\mathrm{op}}$ given by the empty set $\emptyset$ and the disjoint-union tensor $\uplus$.

Note that, as the tensor unit is terminal, the tensor product comes equipped with projections:

$$\mathsf{p}_1 : X \otimes Y \xrightarrow{X \otimes !} X \otimes 1 \xrightarrow{\cong} X \ ,$$
$$\mathsf{p}_2 : X \otimes Y \xrightarrow{! \otimes Y} 1 \otimes Y \xrightarrow{\cong} Y \ .$$

The *presheaf of names* $N \in \mathbf{Set}^{\mathbb{I}}$ is the inclusion of $\mathbb{I}$ into $\mathbf{Set}$.

A π-algebra is an object $A \in \mathbf{Set}^{\mathbb{I}}$ together with operations $\mathsf{nil} : 1 \to A$, $\mathsf{choice} : A^2 \to A$, $\mathsf{out} : N \times N \times A \to A$, $\mathsf{in} : N \times A^N \to A$, $\mathsf{tau} : A \to A$, and $\mathsf{new} : (N \multimap A) \to A$ satisfying the equations of [32, Sections 3.1–3.3 and 3.5]. These algebras, and their homomorphisms, form the category $\mathcal{PI}(\mathbf{Set}^{\mathbb{I}})$.

As mentioned in [32], there is a difficulty in expressing π-algebras as algebras for an enriched algebraic theory. Indeed, the concept of π-algebra relies on the consideration of two enriching structures, but enriched algebraic theories consider only one. More precisely, the operation $\mathsf{new} : (N \multimap A) \to A$ is an operation in $\mathbf{Set}^{\mathbb{I}}$ enriched over itself with respect to the monoidal closed structure; whilst the other operations are operations in $\mathbf{Set}^{\mathbb{I}}$ enriched over itself with respect to the cartesian closed structure. Thus one cannot use enriched algebraic theories to represent π-algebras and thereby establish the existence of free models (*i.e.*, that of a left adjoint to the forgetful functor $U_\pi : \mathcal{PI}(\mathbf{Set}^{\mathbb{I}}) \to \mathbf{Set}^{\mathbb{I}}$ mapping a π-algebra to its carrier object).

As we now proceed to show, the operations and the equations for π-algebras yield a functorial signature together with functorial equations. The functorial signature Σ_π on $\mathbf{Set}^{\mathbb{I}}$ is given by setting

$$\Sigma_\pi(A) = 1 + A^2 + (N \times N \times A) + (N \times A^N) + A + (N \multimap A)\,.$$

In [32], the equations for π-algebras are expressed entirely in the internal language of $\mathbf{Set}^{\mathbb{I}}$ (see also [12]), and hence are shorthand for certain commuting diagrams. One can easily see that these commuting diagrams directly define functorial equations. As an example, we consider the equation establishing the inactivity of a process that inputs on a restricted channel. In the internal language, the equation is given by

$$p : N \multimap A^N \;\vdash\; \mathsf{new}\Big(\nu x : N.\, \mathsf{in}(x\,,\, p\,@\,x)\Big) \;=\; \mathsf{nil} \;:\; A$$

This equation stands for the commutativity of the following diagram:

$$\begin{array}{ccccc}
N \multimap A^N & \xrightarrow{\overline{\langle \mathsf{p}_2, \epsilon^N_{A^N}\rangle}} & N \multimap (N \times A^N) & \xrightarrow{N \multimap (\mathsf{in})} & N \multimap A \\
\downarrow{\scriptstyle !} & & & & \downarrow{\scriptstyle \mathsf{new}} \\
1 & & \xrightarrow{\qquad\quad \mathsf{nil} \qquad\quad} & & A
\end{array}$$

where $\overline{\langle \mathsf{p}_2, \epsilon^N_{A^N}\rangle}$ is the transpose of the map

$$\langle\, \mathsf{p}_2\,,\, \epsilon^N_{A^N}\,\rangle \;:\; (N \multimap A^N) \otimes N \longrightarrow (N \times A^N)\ .$$

The commuting diagram directly yields a parallel pair of functors

$$\Sigma_\pi\text{-}\mathbf{Alg} \rightrightarrows \Big(N \multimap (-)^N\Big)\text{-}\mathbf{Alg}$$

over $\mathbf{Set}^{\mathbb{I}}$.

The functorial signature Σ_π and the functorial equations induced from the axioms of π-algebras constitute an equational system $\mathbb{S}_\pi$ on $\mathbf{Set}^{\mathbb{I}}$ such that $\mathbb{S}_\pi$-$\mathbf{Alg} \cong \mathcal{PI}(\mathbf{Set}^{\mathbb{I}})$. From the fact that the presheaves N and 2 are finitely presentable in $\mathbf{Set}^{\mathbb{I}}$, one can easily see that every endofunctor of $\mathbb{S}_\pi$ is finitary (or equivalently, that it preserves colimits of κ-chains for every infinite limit ordinal κ). Thus the following result follows from Theorem 6.1.

Proposition 7.1 *The category of π-algebras $\mathcal{PI}(\mathbf{Set}^{\mathbb{I}}) \cong \mathbb{S}_\pi$-$\mathbf{Alg}$ is cocomplete and monadic over $\mathbf{Set}^{\mathbb{I}}$ with the induced monad being finitary.*

The above discussion also applies more generally to axiomatic settings as in [12] and, in particular, to π-algebras over nominal sets, $\omega\mathbf{Cpo}^{\mathbb{I}}$, *etc.*

We present the concept of Σ-monoid for an endofunctor Σ with a pointed strength [13,10] and consider it from the point of view of equational systems. The theory of equational systems is then used to provide an explicit description of free Σ-monoids. We then show that, for Σ_λ the functorial signature of the lambda-calculus, the $\beta\eta$ identities are straightforwardly expressible as functorial equations. The theory of equational systems is further used to relate the arising algebraic models by adjunctions.

7.2.1 Σ-monoids

Let Σ be an endofunctor on a monoidal category $\mathscr{C} = (\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$. A *pointed strength* for Σ is a natural transformation

$$\mathsf{st}_{X,(Y,y:I\to Y)} \; : \; \Sigma(X) \otimes Y \dot{\to} \Sigma(X \otimes Y) \; : \; \mathscr{C} \times (I/\mathscr{C}) \to \mathscr{C}$$

satisfying coherence conditions analogous to those of strengths [24]; that is, such that the diagrams

$$\begin{array}{ccc}
\Sigma(A) \otimes I & \xrightarrow{\mathsf{st}_{A,(I,\mathrm{id}_I:I\to I)}} & \Sigma(A \otimes I) \\
 & \underset{\rho_{\Sigma(A)}}{\overset{\cong}{\searrow}} & \Big\downarrow {\scriptstyle \cong\ \Sigma(\rho_A)} \\
 & & \Sigma(A)
\end{array}$$

$$\begin{array}{ccccc}
(\Sigma(A) \otimes B) \otimes C & \xrightarrow{\mathsf{st}_{A,(B,b:I\to B)}\otimes C} & \Sigma(A \otimes B) \otimes C & \xrightarrow{\mathsf{st}_{A\otimes B,(C,c:I\to C)}} & \Sigma((A \otimes B) \otimes C) \\
\Big\downarrow {\scriptstyle \cong\ \alpha_{\Sigma(A),B,C}} & & & & \Big\downarrow {\scriptstyle \cong\ \Sigma(\alpha_{A,B,C})} \\
\Sigma(A) \otimes (B \otimes C) & & \xrightarrow{\mathsf{st}_{A,(B\otimes C,(b\otimes c)\circ\rho_I^{-1}:I\to B\otimes C)}} & & \Sigma(A \otimes (B \otimes C))
\end{array}$$

commute for all $A \in \mathscr{C}$ and $(B, b : I \to B), (C, c : I \to C) \in I/\mathscr{C}$.

Remark *The notion of pointed strength arises as a special case of that of a strength for an action of a monoidal category on a category (see [10] and also [22]).*

For an endofunctor Σ with a pointed strength st on a monoidal category $\mathscr{C}$, the category of Σ-*monoids* Σ-$\mathbf{Mon}(\mathscr{C})$ has objects given by quadruples (X, s, m, e) where $(X, s : \Sigma X \to X)$ is a Σ-algebra and $(X, m : X \otimes X \to X, e : I \to X)$ is a monoid in $\mathscr{C}$ satisfying the compatibility law requiring that the diagram

$$\begin{array}{ccccc}
\Sigma(X) \otimes X & \xrightarrow{\mathsf{st}_{X,(X,e:I\to X)}} & \Sigma(X \otimes X) & \xrightarrow{\Sigma(m)} & \Sigma(X) \\
\Big\downarrow {\scriptstyle s\otimes X} & & & & \Big\downarrow {\scriptstyle s} \\
X \otimes X & & \xrightarrow{m} & & X
\end{array}$$

commutes; morphisms are maps of $\mathscr{C}$ that are both Σ-algebra and monoid homomorphisms.

7.2.2 Equational system for Σ-monoids

There are problems in presenting Σ-monoids as algebras for an enriched algebraic theory. For one thing, if the monoidal category $\mathscr{C}$ is not closed, then $\mathscr{C}$ is not enriched over itself. More importantly, however, even when $\mathscr{C}$ is closed, the operation $m : X \otimes X \to X$ is not directly expressible as an operation of an enriched algebraic theory. Equational systems overcome these problems.

Let $\mathscr{C} = (\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category with binary coproducts. For an endofunctor Σ on $\mathscr{C}$ with a pointed strength st, the equational system $\mathbb{M}_\Sigma$ of Σ-monoids is defined as

$$(\mathscr{C} : F_\Sigma \rhd G_\Sigma \vdash L_\Sigma = R_\Sigma)$$

with

$$F_\Sigma(X) \;=\; \Sigma(X) + (X \otimes X) + I$$
$$G_\Sigma(X) \;=\; \big((X \otimes X) \otimes X\big) + (I \otimes X) + (X \otimes I) + (\Sigma(X) \otimes X)$$
$$L_\Sigma(\, X, [s, m, e]\,)$$
$$= (X, [\quad m \circ (m \otimes \mathrm{id}_X) \quad , \quad \lambda_X \quad , \quad \rho_X \quad , \quad m \circ (s \otimes \mathrm{id}_X) \quad])$$
$$R_\Sigma(\, X, [s, m, e]\,)$$
$$= (X, [\, m \circ (\mathrm{id}_X \otimes m) \circ \alpha_{X,X,X} \,, m \circ (e \otimes \mathrm{id}_X) \,, m \circ (\mathrm{id}_X \otimes e) \,,\, s \circ \Sigma(m) \circ \mathsf{st}_{X,(X,e)} \,])$$

The functoriality of L_Σ and R_Σ follows from the naturality of α, λ, ρ, and st. By construction, $\mathbb{M}_\Sigma$-**Alg** is (isomorphic to) Σ-**Mon**($\mathscr{C}$).

7.2.3 Free Σ-monoids

We now proceed to apply the theory of equational systems developed in this paper to the algebra of Σ-monoids. For instance, by Theorems 4.6 and 4.8, if $\mathscr{C}$ is cocomplete, and the endofunctor $\Sigma : \mathscr{C} \to \mathscr{C}$ and the tensor product $\otimes : \mathscr{C}^2 \to \mathscr{C}$ preserve epimorphisms and colimits of ω-chains, then the category Σ-**Mon**($\mathscr{C}$) is monadic over $\mathscr{C}$, and free Σ-monoids on objects in $\mathscr{C}$ can be constructed as in diagram (2) followed by (4).

While this provides an abstract construction of free Σ-monoids, when the monoidal structure is closed, one can go further and give an explicit description of free Σ-monoids by exploiting the following fact.

When $\mathscr{C}$ is monoidal closed, if the initial $(I + \Sigma(-))$-algebra $\mu X.\, I + \Sigma X$ exists, then the initial Σ-monoid exists and has carrier object $\mu X.\, I + \Sigma X$ equipped with an appropriate Σ-monoid structure (see [13]).

Indeed, a free Σ-monoid over $A \in \mathscr{C}$ is an initial $\mathbb{M}^A_\Sigma$-algebra for $\mathbb{M}^A_\Sigma$ the equational system

$$\Big(\mathscr{C} : (A + F_\Sigma(-)) \rhd G_\Sigma \vdash L_\Sigma\, U_A = R_\Sigma\, U_A\Big) \ ,$$

where U_A denotes the forgetful functor $\big(A + F_\Sigma(-)\big)$-**Alg** $\to F_\Sigma$-**Alg**. Furthermore, for the endofunctor $(A \otimes -) + \Sigma(-)$ on $\mathscr{C}$ with the pointed strength given by the composite

$$\begin{aligned}&\big((A \otimes X) + \Sigma(X)\big) \otimes Y \\ &\quad \cong \ \big((A \otimes X) \otimes Y\big) + \Sigma(X) \otimes Y \xrightarrow{\alpha_{A,X,Y} + \mathsf{st}_{X,(Y,y)}} \big(A \otimes (X \otimes Y)\big) + \Sigma(X \otimes Y) \ ,\end{aligned}$$

one can establish the isomorphism $p : \mathbb{M}^A_\Sigma\text{-}\mathbf{Alg} \cong \mathbb{M}_{(A\otimes -)+\Sigma(-)}\text{-}\mathbf{Alg} : q$ with p and q given by

$$\begin{aligned}&p(X,\ [a,s,m,e] \,:\, A \,+\, \Sigma X \,+\, X \otimes X \,+\, I \,\longrightarrow\, X) \\ &\quad = \ (X,\ [m \circ (a \otimes \mathrm{id}_X), s, m, e] \,:\, A \otimes X \,+\, \Sigma X \,+\, X \otimes X \,+\, I \,\longrightarrow\, X)\end{aligned}$$

$$\begin{aligned}&q(X,\ [b,s,m,e] \,:\, A \otimes X \,+\, \Sigma X \,+\, X \otimes X \,+\, I \,\longrightarrow\, X) \\ &\quad = \ (X,\ [b \circ (\mathrm{id}_A \otimes e) \circ \rho_A^{-1}, s, m, e] \,:\, A \,+\, \Sigma X \,+\, X \otimes X \,+\, I \,\longrightarrow\, X)\ .\end{aligned}$$

Thus, we have the following result (see also [10]).

Proposition 7.2 *Let $\mathscr{C}$ be a monoidal closed category with binary coproducts. For any object $A \in \mathscr{C}$, if the initial $\big(I + (A \otimes -) + \Sigma(-)\big)$-algebra $\mu X.\, I + A \otimes X + \Sigma X$ exists, then the free Σ-monoid on A exists and has carrier object $\mu X.\, I + A \otimes X + \Sigma X$ equipped with an appropriate Σ-monoid structure.*

7.2.4 Lambda-calculus algebras

As a concrete example of algebras with monoid structure, we start by considering the syntax of the λ-calculus, with models given as certain Σ_λ-monoids on the presheaf category $\mathbf{Set}^{\mathbb{F}}$ for $\mathbb{F}$ the (essentially small) category of finite sets and functions.

We quickly review the structure of $\mathbf{Set}^{\mathbb{F}}$ needed here. Besides the cartesian closed structure, the presheaf category $\mathbf{Set}^{\mathbb{F}}$ is equipped with the substitution monoidal structure $(V, \bullet)$, where the unit V is the embedding of $\mathbb{F}$ into $\mathbf{Set}$

and the tensor $\bullet$ is given by the coend formula

$$(X \bullet Y)(n) = \int^{k \in \mathbb{F}} X(k) \times (Yn)^k \ .$$

This substitution monoidal structure is closed.

The endofunctor $(-)^V$ on $\mathbf{Set}^{\mathbb{F}}$ has the property of *shifting* presheaves; in that, for any presheaf $X \in \mathbf{Set}^{\mathbb{F}}$, the set $X^V(n)$ can be presented as $X(n+1)$ for all finite sets $n \in \mathbb{F}$.

A λ-*prealgebra* [13] is a Σ_λ-monoid for the endofunctor $\Sigma_\lambda X = X^V + X^2$ with a suitable pointed strength on the presheaf category $\mathbf{Set}^{\mathbb{F}}$. The operations of a Σ_λ-monoid

$$(X, [\mathsf{abs}, \mathsf{app}, \mathsf{sub}, \mathsf{var}] : X^V + X^2 + (X \bullet X) + V \longrightarrow X)$$

provide interpretations of λ-abstraction ($\mathsf{abs} : X^V \to X$), application ($\mathsf{app} : X^2 \to X$), capture-avoiding simultaneous substitution ($\mathsf{sub} : X \bullet X \to X$), and variables ($\mathsf{var} : V \to X$).

The initial Σ_λ-monoid has carrier object $\mu X.\, V + X^V + X^2$. It consists of α-equivalence classes of λ-terms with variables from V, and thus provides an abstract notion of syntax for the λ-calculus (see [13]). The syntactic description of free Σ_λ-monoids has been considered in [20,10].

The $\beta\eta$ identities for a λ-prealgebra on X are expressed, in the internal language, by the following equations

$$(\beta) \quad f : X^V, x : X \ \vdash \ \mathsf{app}(\mathsf{abs}(f), x) \ = \ \mathsf{sub}(f\langle x \rangle) \ : \ X$$

$$(\eta) \quad x : X \ \vdash \ \mathsf{abs}\big(\lambda v : V.\, \mathsf{app}(x, \mathsf{var}\, v)\big) \ = \ x \ : \ X$$

where the map $-\langle = \rangle : X^V \times X \to X \bullet X$ embeds $X^V \times X$ into $X \bullet X$. Indeed, the equations stand for the following commuting diagrams

$$\begin{array}{ccc} X^V \times X & \xrightarrow{-\langle = \rangle} & X \bullet X \\ {\scriptstyle \mathsf{abs} \times X} \downarrow & & \downarrow {\scriptstyle \mathsf{sub}} \\ X \times X & \xrightarrow{\mathsf{app}} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{x:X \,\vdash\, \lambda v:V.\, \mathsf{app}(x, \mathsf{var}\, v)} & X^V \\ & {\scriptstyle \mathrm{id}_X} \searrow & \downarrow {\scriptstyle \mathsf{abs}} \\ & & X \end{array}$$

where the map $x : X \ \vdash \ \lambda v : V.\, \mathsf{app}(x, \mathsf{var}\, v)$ is the transpose of the composite

$$X \times V \xrightarrow{X \times \mathsf{var}} X \times X \xrightarrow{\mathsf{app}} X \ .$$

These commuting diagrams provide a functorial equation

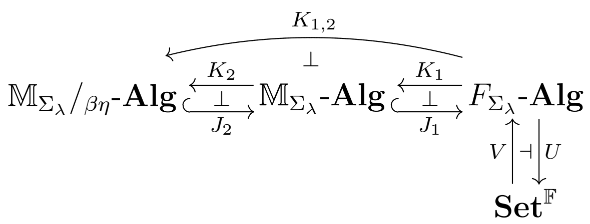
$$L_{\beta\eta} = R_{\beta\eta} : F_{\Sigma_\lambda}\text{-}\mathbf{Alg} \to G_{\beta\eta}\text{-}\mathbf{Alg} \ ,$$

for $G_{\beta\eta}X = (X^V \times X) + X$, and yield the equational system of λ-*algebras*

$$\mathbb{M}_{\Sigma_\lambda}/_{\beta\eta} \;=\; \Big(\mathbf{Set}^{\mathbb{F}} : F_{\Sigma_\lambda} \rhd (G_{\Sigma_\lambda} + G_{\beta\eta}) \vdash [L_{\Sigma_\lambda}, L_{\beta\eta}] = [R_{\Sigma_\lambda}, R_{\beta\eta}]\Big)$$

from that of λ-prealgebras $\mathbb{M}_{\Sigma_\lambda} = (\mathbf{Set}^{\mathbb{F}} : F_{\Sigma_\lambda} \rhd G_{\Sigma_\lambda} \vdash L_{\Sigma_\lambda} = R_{\Sigma_\lambda})$.

From the coend formula for the substitution tensor and the fact that in the category of sets filtered colimits commute with finite limits, it follows that $\bullet : \mathbf{Set}^{\mathbb{F}} \times \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$ preserves filtered colimits, and it is also easily seen that it preserves epimorphisms. Furthermore, also the endofunctors $(-)^V$ and $(-)^2$ preserve filtered colimits and epimorphisms. Hence, so do the endofunctors F_{Σ_λ}, G_{Σ_λ}, and $G_{\beta\eta}$. Thus, from one application of Theorem 4.6 and two applications of Theorem 4.8, we obtain the adjunctions $V \dashv U$, $K_1 \dashv J_1$, and $K_{1,2} \dashv J_1\, J_2$



and consequently have that $K_2 = K_{1,2}\, J_1 \dashv J_2$ as in the diagram above.

Moreover, by examining the construction (4) of the free $\mathbb{M}_{\Sigma_\lambda}/_{\beta\eta}$-algebra over the initial $\mathbb{M}_{\Sigma_\lambda}$-algebra along K_2, one sees that the presheaf of α-equivalence classes of λ-terms is first quotiented by the $\beta\eta$ identities, and then by the congruence rules for the operations abs, app, and sub. It follows that the initial $\mathbb{M}_{\Sigma_\lambda}/_{\beta\eta}$-algebra is the presheaf of $\alpha\beta\eta$-equivalence classes of λ-terms.

7.3 Nominal equational theories

Clouston and Pitts [7] have recently introduced Nominal Equational Logic (NEL) as an extension of equational logic with names and assertions on their freshness. We show in this section that every *NEL theory* can be represented as an equational system in the sense that the respective categories of algebras coincide. By further showing that the equational system representation satisfies the hypothesis of Theorems 4.7 and 4.8, the monadicity and cocompleteness of categories of algebras for NEL theories follows (see Corollary 6.6). We also give explicit descriptions of free algebras for NEL theories as derived from their inductive construction. For brevity, we only consider the single-sorted case; the multi-sorted one being treated analogously. All these results can be also seen to apply to the *nominal algebras* of Gabbay and Mathijssen [15]. However, we do not dwell on this here.

7.3.1 Nominal sets

For a fixed countably infinite set A of atoms, the group $\mathfrak{S}_0(\mathsf{A})$ of finite permutations of atoms consists of the bijections on A that fix all but finitely many elements of A. A $\mathfrak{S}_0(\mathsf{A})$-*action* $X = (|X|, \cdot)$ consists of a set $|X|$ equipped with a function $- \cdot = : \mathfrak{S}_0(\mathsf{A}) \times |X| \to |X|$ such that $\mathrm{id}_{\mathsf{A}} \cdot x = x$ and $\pi' \cdot (\pi \cdot x) = (\pi'\pi) \cdot x$ for all $x \in |X|$ and $\pi, \pi' \in \mathfrak{S}_0(\mathsf{A})$. $\mathfrak{S}_0(\mathsf{A})$-actions form a category with morphisms $X \to Y$ given by *equivariant* functions; that is, functions $f : |X| \to |Y|$ such that $f(\pi \cdot x) = \pi \cdot (fx)$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$ and $x \in |X|$.

By an element x of a $\mathfrak{S}_0(\mathsf{A})$-action X, denoted $x \in X$, we mean that x is a member of $|X|$. For a $\mathfrak{S}_0(\mathsf{A})$-action X, a finite subset S of A is said to *support* $x \in X$ if for all atoms $a, a' \notin S$, $(a\,a') \cdot x = x$, where the *transposition* $(a\,a')$ is the bijection that swaps a and a'. A *nominal set* is a $\mathfrak{S}_0(\mathsf{A})$-action in which every element has finite support. As an example, note that the set of atoms A becomes the *nominal set of atoms* $\mathbb{A}$ when equipped with the evaluation action $\pi \cdot a = \pi(a)$. A further example is given by $\mathfrak{S}_0(\mathsf{A})$ equipped with the conjugation action $\pi \cdot \sigma = \pi\sigma\pi^{-1}$, which we denote as $\mathfrak{S}_0(\mathbb{A})$.

The supports of an element of a nominal set are closed under intersection, and we write $\mathsf{supp}_X(x)$, or simply $\mathsf{supp}(x)$ when X is clear from the context, for the intersection of the supports of x in the nominal set X. For instance, $\mathsf{supp}_{\mathbb{A}}(a) = \{ a \}$ and $\mathsf{supp}_{\mathfrak{S}_0(\mathbb{A})}(\sigma) = \{ a \in \mathsf{A} \mid \sigma(a) \neq a \}$. For elements x and y of two, possibly distinct, nominal sets X and Y, we write $x \# y$ whenever $\mathsf{supp}_X(x)$ and $\mathsf{supp}_Y(y)$ are disjoint. Thus, for $a \in \mathbb{A}$ and $x \in X$, $a \# x$ stands for $a \notin \mathsf{supp}_X(x)$; that is, *$a$ is fresh for x*.

For an element x of a nominal set X, and $\pi, \pi' \in \mathfrak{S}_0(\mathsf{A})$ such that $\pi(a) = \pi'(a)$ for all $a \in \mathsf{supp}(x)$, we have that $\pi \cdot x = \pi' \cdot x$. Thus, for a finite set of atoms $S \supseteq \mathsf{supp}(x)$ and an injective function $\alpha : S \rightarrowtail \mathsf{A}$ it makes sense to define $\alpha \cdot x$ as $\widetilde{\alpha} \cdot x$ for $\widetilde{\alpha} \in \mathfrak{S}_0(\mathsf{A})$ any permutation extending α.

We let **Nom** be the full subcategory of the category of $\mathfrak{S}_0(\mathsf{A})$-actions consisting of nominal sets, and briefly consider its structure relevant to us here.

The *coproduct* $\coprod_{k\in K} X_k$ of a family of nominal sets $\{ X_k \}_{k \in K}$ is the nominal set with $|\coprod_{k\in K} X_k| = \coprod_{k \in K} |X_k|$ and action $\pi \cdot (k, x) = (k, \pi \cdot x)$. As usual we write $X_1 + \cdots + X_n$ for $\coprod_{k \in \{1,\ldots,n\}} X_k$.

The *product* $\prod_{i\in I} X_i$ of a *finite* family of nominal sets $\{ X_i \}_{i\in I}$ is the nominal set with $|\prod_{i \in I} X_i| = \prod_{i\in I} |X_i|$ and action $\pi \cdot (x_i)_{i\in I} = (\pi \cdot x_i)_{i \in I}$. As usual we write $X_1 \times \cdots \times X_n$ for $\prod_{i\in\{1,\ldots,n\}} X_i$, and X^I for $\prod_{i\in I} X$. More generally, the *exponential* X^Y of nominal sets X and Y consists of all finitely supported functions $|Y| \to |X|$ with respect to the action that inversely acts on the input and directly acts on the output; *i.e.*, $\pi \cdot f = \lambda x.\, \pi \cdot f(\pi^{-1} \cdot x)$.

The *separating tensor* $\#_{i\in I} X_i$ of a *finite* family of nominal sets $\{ X_i \}_{i\in I}$ is the subnominal set of $\prod_{i\in I} X_i$ with underlying set given by

$$\{ (x_i)_{i\in I} \mid x_i \# x_j \text{ for all } i \neq j \} .$$

We write $X_1 \# \cdots \# X_n$ for $\#_{i\in\{1,\ldots,n\}} X_i$, and $X^{\# I}$ for $\#_{i\in I} X$. Thus, for instance, the nominal set $\mathbb{A}^{\# I}$ for a finite set I is the subset of A^I consisting of the injections $\imath : I \rightarrowtail \mathsf{A}$ with action given by post-composition; *i.e.*, $\pi \cdot \imath = \pi\imath$. The separating tensor carries a symmetric monoidal closed structure.

For every set S, we define two nominal sets $\underline{S}$ and $\overline{S}$: the nominal set $\underline{S}$ has underlying set S and projection action $\pi \cdot s = s$; the nominal set $\overline{S}$ is the product $\mathfrak{S}_0(\mathbb{A}) \times \underline{S}$.

For a nominal set X, the nominal set $\mathcal{P}_0(X)$ has underlying set $\mathcal{P}_0|X|$, the set of finite subsets of $|X|$, and pointwise action $\pi \cdot S = \{ \pi \cdot x \mid x \in S \}$. In particular, $\mathcal{P}_0(\underline{S}) = \underline{\mathcal{P}_0(S)}$ for every set S. Note also that, for $A \in \mathcal{P}_0(\mathbb{A})$ and $x \in X$, $A \# x$ stands for $a \# x$ for all $a \in A$.

7.3.2 NEL theories

A NEL theory consists of a signature defining its operators together with the set of axioms that these should obey.

A *NEL signature* Σ is specified by a family of nominal sets $\{ \Sigma(n) \}_{n\in\mathbb{N}}$, each of which consists of *operators* of arity $n \in \mathbb{N}$.

The *nominal set of terms* $T_\Sigma(V)$ on a nominal set V is inductively defined by the following rules:

$$\frac{v \in V}{v \in T_\Sigma(V)} \qquad \frac{t_i \in T_\Sigma(V) \quad (i = 1, \ldots, n)}{\mathsf{o}\ t_1 \ldots t_n \in T_\Sigma(V)} \ (\mathsf{o} \in \Sigma(n)) \tag{6}$$

and equipped with the action inductively defined by:

$$\begin{aligned} \pi \cdot_{T_\Sigma(V)} v &= \pi \cdot_V v \\ \pi \cdot_{T_\Sigma(V)} (\mathsf{o}\ t_1 \ldots t_n) &= (\pi \cdot_{\Sigma(n)} \mathsf{o})\ (\pi \cdot_{T_\Sigma(V)} t_1) \ldots (\pi \cdot_{T_\Sigma(V)} t_n) \qquad (\mathsf{o} \in \Sigma(n)) \end{aligned}$$

We now fix a countably infinite set V of variables. The nominal set of *freshness contexts* is defined as

$$\coprod_{S\in\mathcal{P}_0(\mathsf{V})} \left(\mathcal{P}_0\mathbb{A} \blacktriangleright (\mathcal{P}_0\mathbb{A})^S \right)$$

where $X \blacktriangleright Y$ denotes the subnominal set of $X \times Y$ with underlying set $\{ (x, y) \in |X| \times |Y| \mid \mathsf{supp}(x) \supseteq \mathsf{supp}(y) \}$. Thus, the nominal set of freshness contexts has elements $\nabla = (|\nabla|, \nabla^{\mathcal{A}}, \nabla^{\#})$ given by a finite set of variables $|\nabla| \subset \mathsf{V}$, a finite set of atoms $\nabla^{\mathcal{A}} \subset \mathsf{A}$, and a function $\nabla^{\#} : |\nabla| \to \mathcal{P}_0(\nabla^{\mathcal{A}})$ with the following action

$$\pi \cdot (|\nabla|, \nabla^{\mathcal{A}}, \nabla^{\#}) = (\, |\nabla| \,,\, \pi \cdot_{\mathcal{P}_0(\mathbb{A})} \nabla^{\mathcal{A}} \,,\, \lambda x \in |\nabla|.\, \pi \cdot_{\mathcal{P}_0(\mathbb{A})} \nabla^{\#}(x) \,) \ .$$

Note that $\mathsf{supp}(\nabla) = \nabla^{\mathcal{A}}$.

If $|\nabla| = \{ x_1, \ldots, x_n \}$, $\nabla^{\mathcal{A}} = \{ a_1, \ldots, a_m \}$, and $\nabla^{\#}(x_i) = A_i$ for $i = 1, \ldots, n$, we write ∇ as

$$a_1, \ldots, a_m \blacktriangleright A_1 \# x_1, \ldots, A_n \# x_n$$

where we also abbreviate $\emptyset \# x$ as x and $\{a\} \# x$ as $a \# x$.

By a term t in a freshness context ∇, written $\nabla \vdash t$, we mean $t \in T_\Sigma(\,\overline{|\nabla|}\,)$ such that $\mathsf{supp}(t) \subseteq \nabla^{\mathcal{A}}$. That is, the grammar for terms in freshness contexts is as follows:

$$\begin{aligned} t ::= {} & \sigma\, x && (\sigma \in \mathfrak{S}_0(\mathsf{A}) \text{ with } \mathsf{supp}(\sigma) \subseteq \nabla^{\mathcal{A}},\ x \in |\nabla|) \\ \mid {} & \mathsf{o}\ t_1 \ldots t_n && (\mathsf{o} \in \Sigma(n) \text{ with } \mathsf{supp}(\mathsf{o}) \subseteq \nabla^{\mathcal{A}}) \end{aligned}$$

where we use the notational convention of abbreviating (σ, x) as $\sigma\, x$ and further abbreviating this as x when σ is the identity. Note that $\nabla \vdash t$ implies $\pi \cdot \nabla \vdash \pi \cdot t$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$.

A *NEL theory* is given by a NEL signature Σ together with a set of *axioms* consisting of judgements of the form

$$\nabla \vdash t \approx t'$$

where t and t' are terms in the freshness context ∇.

We give the canonical example of NEL theory. The NEL signature Σ_λ for the untyped λ-calculus [7] (see also [15]) is given by the nominal sets of operators

$$\Sigma_\lambda(0) = \{\, \boldsymbol{V}_a \mid a \in \mathsf{A} \,\} \ , \quad \Sigma_\lambda(1) = \{\, \boldsymbol{L}_a \mid a \in \mathsf{A} \,\} \ , \quad \Sigma_\lambda(2) = \{\, \boldsymbol{A} \,\} \ ,$$
$$\Sigma_\lambda(n) = \emptyset \quad (n \geq 3)$$

with actions

$$\pi \cdot \boldsymbol{V}_a = \boldsymbol{V}_{\pi(a)} \ , \quad \pi \cdot \boldsymbol{L}_a = \boldsymbol{L}_{\pi(a)} \ , \quad \pi \cdot \boldsymbol{A} = \boldsymbol{A} \ .$$

The NEL theory for $\alpha\beta\eta$-equivalence of untyped λ-terms consists of the following axioms.

$$a, a' \blacktriangleright a' \mathrel{\#\!\!\!/} x \vdash \boldsymbol{L}_a\, x \approx \boldsymbol{L}_{a'}\, ((a\, a')\, x) \tag{α}$$

$$a \blacktriangleright a \mathrel{\#\!\!\!/} x,\ x' \vdash \boldsymbol{A}\, (\boldsymbol{L}_a\, x)\, x' \approx x \tag{β-1}$$

$$a \blacktriangleright x' \vdash \boldsymbol{A}\, (\boldsymbol{L}_a\, \boldsymbol{V}_a)\, x' \approx x' \tag{β-2}$$

$$a, a' \blacktriangleright x,\ a' \mathrel{\#\!\!\!/} x' \vdash \boldsymbol{A}\, (\boldsymbol{L}_a\, (\boldsymbol{L}_{a'}\, x))\, x' \approx \boldsymbol{L}_{a'}\, (\boldsymbol{A}\, (\boldsymbol{L}_a\, x)\, x') \tag{β-3}$$

$$a \blacktriangleright x_1,\ x_2,\ x' \vdash \boldsymbol{A}\, (\boldsymbol{L}_a\, (\boldsymbol{A}\, x_1\, x_2))\, x' \approx \boldsymbol{A}\, (\boldsymbol{A}\, (\boldsymbol{L}_a\, x_1)\, x')\, (\boldsymbol{A}\, (\boldsymbol{L}_a\, x_2)\, x') \tag{β-4}$$

$$a, a' \blacktriangleright a' \mathrel{\#\!\!\!/} x \vdash \boldsymbol{A}\, (\boldsymbol{L}_a\, x)\, \boldsymbol{V}_{a'} \approx (a\, a')\, x \tag{β-5}$$

$$a \blacktriangleright a \mathrel{\#\!\!\!/} x \vdash x \approx \boldsymbol{L}_a\, (\boldsymbol{A}\, x\, \boldsymbol{V}_a) \tag{η}$$

Remark *The work reported in [7] is based on judgements of the form*

$$\nabla \vdash A \mathrel{\#\!\!\!/} t \approx t'$$

where A is a finite set of atoms that imposes name freshness conditions on the terms of the equation. However, Clouston has shown that this extension, though convenient, does not add expressive power; as every such axiom can be equivalently encoded as one without freshness conditions (see also [15, Theorem 5.5]). For instance, the α-equivalence axiom above is the encoding of the following one

$$a \blacktriangleright x \vdash \{a\} \mathrel{\#\!\!\!/} \boldsymbol{L}_a\, x \approx \boldsymbol{L}_a\, x\ .$$

A Σ-*structure* (M, e) for a NEL signature Σ is given by a nominal set M and an $\mathbb{N}$-indexed family e of equivariant functions $\mathsf{e}_n : \Sigma(n) \times M^n \to M$, referred to as *evaluation functions*. The evaluation functions extend from operators to terms to give the equivariant function $\bar{\mathsf{e}}_V : T_\Sigma(V) \times M^V \to M$, for each nominal set V, inductively defined by:

$$\begin{aligned} \bar{\mathsf{e}}_V(v, m) &= m(v)\ , \\ \bar{\mathsf{e}}_V(\mathsf{o}\ t_1 \ldots t_n,\ m) &= \mathsf{e}_n(\mathsf{o},\ \bar{\mathsf{e}}_V(t_1, m),\ \ldots,\ \bar{\mathsf{e}}_V(t_n, m))\ . \end{aligned}$$

By a *valuation* m of a freshness context ∇ in a nominal set M, we mean $m \in M^{|\nabla|}$ such that $\nabla^{\#}(x) \mathrel{\#} m(x)$ for all $x \in |\nabla|$. It follows that $\pi \cdot m$ is a valuation of $\pi \cdot \nabla$ in M for all $\pi \in \mathfrak{S}_0(\mathsf{A})$. For every valuation m of ∇, the function $\overline{m} : \overline{|\nabla|} \to M$ defined by setting $\overline{m}(\pi, x) = \pi \cdot m(x)$ is finitely supported with $\mathsf{supp}(m) = \bigcup_{x \in |\nabla|} \mathsf{supp}(m(x))$ and hence provides an extension $\overline{m} \in M^{\overline{|\nabla|}}$ of $m \in M^{|\nabla|}$.

A Σ-structure (M, e) is said to *satisfy* the judgement $\nabla \vdash t \approx t'$ if

$$\bar{\mathsf{e}}_{\overline{|\nabla|}}(t, \overline{m}) = \bar{\mathsf{e}}_{\overline{|\nabla|}}(t', \overline{m})$$

for all valuations m of ∇ in M.

A $\mathbb{T}$-*algebra* for a NEL theory $\mathbb{T} = (\Sigma, E)$ is a Σ-structure that satisfies every axiom in E. A *homomorphism* from a $\mathbb{T}$-algebra (M, e) to another one (M', e') is an equivariant function $h : M \to M'$ such that

$$h(\mathsf{e}_n(\mathsf{o}, m_1, \ldots, m_n)) = \mathsf{e}'_n(\mathsf{o}, h(m_1), \ldots, h(m_n))$$

for all $n \in \mathbb{N}$, $\mathsf{o} \in \Sigma(n)$, and $m_1, \ldots, m_n \in M$. $\mathbb{T}$-algebras and homomorphisms form the category $\mathbb{T}$-**Alg**.

7.3.3 NEL theories as equational systems

We will now present every NEL theory $\mathbb{T} = (\Sigma, E)$ as an equational system $\widetilde{\mathbb{T}} = (\mathbf{Nom} : \widetilde{\Sigma} \rhd \widetilde{\Gamma} \vdash \widetilde{L} = \widetilde{R})$ in such a way that the respective categories of algebras are isomorphic.

The *functorial signature* $\widetilde{\Sigma}$ is simply defined as

$$\widetilde{\Sigma}(M) = \coprod_{n \in \mathbb{N}} \Sigma(n) \times M^n \ ,$$

so that $\widetilde{\Sigma}$-algebras and Σ-structures are in bijective correspondence.

Turning the set of axioms into a functorial equation is more involved. We consider first the definition of the functorial context associated to a freshness context. To this end, note that if a Σ-structure satisfies the axiom $\nabla \vdash t \approx t'$ then, by equivariance of the evaluation functions, it also satisfies the judgement $(\pi \cdot \nabla) \vdash (\pi \cdot t) \approx (\pi \cdot t')$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$ (see [7]). Hence the atoms in $\nabla^{\mathcal{A}}$ for the freshness context ∇ of a judgement can be conceptually understood as atom place-holders (or meta-atoms). It follows that the functorial contexts of freshness contexts should be given by a consistent interpretation of both atoms and term variables. This is formalized by defining the *functorial context* Γ_∇ on **Nom** of a freshness context ∇ as

$$\Gamma_\nabla(M) = \{\, (\alpha, m) \in \mathbb{A}^{\#\nabla^{\mathcal{A}}} \times M^{|\nabla|} \mid m \text{ is a valuation of } \alpha \cdot \nabla \text{ in } M \,\} \ .$$

Note that $\alpha \cdot \nabla$, which stands for $\widetilde{\alpha} \cdot \nabla$ where $\widetilde{\alpha} \in \mathfrak{S}_0(\mathsf{A})$ is any permutation extending $\alpha : \nabla^{\mathcal{A}} \rightarrowtail \mathsf{A}$, is well defined because $\nabla^{\mathcal{A}}$ is the support of ∇. The above definition makes Γ_∇ into a functor because for $(\alpha, m) \in \Gamma_\nabla(M)$ and an equivariant function $f : M \to N$, we have that $\mathsf{supp}(f(m(x))) \subseteq \mathsf{supp}(m(x))$ for all $x \in M$ and hence that $f \circ m \in N^{|\nabla|}$ is a valuation of $\alpha \cdot \nabla$ in N, that is, $(\alpha, f \circ m) \in \Gamma_\nabla(N)$.

For a term in a freshness context $\nabla \vdash t$, the *functorial term*

$$F_{\nabla \vdash t} : \widetilde{\Sigma}\text{-}\mathbf{Alg} \to \Gamma_\nabla\text{-}\mathbf{Alg}$$

then maps (M, e) to

$$F_{\nabla \vdash t}(M, \mathsf{e}) : \ \Gamma_\nabla(M) \to M : \ (\alpha, m) \mapsto \overline{\mathsf{e}}_{\overline{|\nabla|}}(\alpha \cdot t, \overline{m}) \ .$$

Note that $\alpha \cdot t$, which stands for $\widetilde{\alpha} \cdot t$ where $\widetilde{\alpha} \in \mathfrak{S}_0(\mathsf{A})$ is any permutation extending $\alpha : \nabla^{\mathcal{A}} \rightarrowtail \mathsf{A}$, is well defined because $\nabla^{\mathcal{A}}$ includes the support of t.

The equivariance of $F_{\nabla \vdash t}(M, \mathsf{e})$ is established as follows:

$$\begin{aligned}
F_{\nabla \vdash t}(M, \mathsf{e})\Big(\pi \cdot (\alpha, m)\Big) &= F_{\nabla \vdash t}(M, \mathsf{e})(\pi\alpha, \pi \cdot m) \\
&= \overline{\mathsf{e}}_{\overline{|\nabla|}}((\pi\alpha) \cdot t, \overline{\pi \cdot m}) \\
&= \overline{\mathsf{e}}_{\overline{|\nabla|}}(\pi \cdot (\alpha \cdot t), \pi \cdot \overline{m}) \\
&= \pi \cdot \overline{\mathsf{e}}_{\overline{|\nabla|}}(\alpha \cdot t, \overline{m}) \\
&= \pi \cdot F_{\nabla \vdash t}(M, \mathsf{e})(\alpha, m)
\end{aligned}$$

where the third identity follows because any extension $\widetilde{\alpha}$ of α makes $\pi\widetilde{\alpha}$ into an extension of $\pi\alpha$, and because $\overline{\pi \cdot m} = \pi \cdot \overline{m}$.

The equational system $\widetilde{\mathbb{T}} = (\mathbf{Nom} : \widetilde{\Sigma} \rhd \widetilde{\Gamma} \vdash \widetilde{L} = \widetilde{R})$ associated to the NEL theory $\mathbb{T} = (\Sigma, E)$ is thus defined as

$$\begin{aligned}
\widetilde{\Sigma} &= \textstyle\coprod_{n \in \mathbb{N}} \Sigma(n) \times (-)^n \ , &\widetilde{\Gamma} &= \textstyle\coprod_{(\nabla \vdash t \approx t') \in E} \Gamma_\nabla \\
\widetilde{L} &= \Big[F_{\nabla \vdash t}\Big]_{(\nabla \vdash t \approx t') \in E} \ , &\widetilde{R} &= \Big[F_{\nabla \vdash t'}\Big]_{(\nabla \vdash t \approx t') \in E}
\end{aligned}$$

Theorem 7.3 *The categories* $\mathbb{T}$-**Alg** *and* $\widetilde{\mathbb{T}}$-**Alg** *are isomorphic.*

PROOF. We prove that a Σ-structure (M, e) satisfies the judgement $\nabla \vdash t \approx t'$ if and only if $F_{\nabla \vdash t}(M, [\mathsf{e}_n]_{n \in \mathbb{N}}) = F_{\nabla \vdash t'}(M, [\mathsf{e}_n]_{n \in \mathbb{N}})$.

The if part is easily shown by considering the inclusion function $\iota \in \mathsf{A}^{\# \nabla^{\mathcal{A}}}$. Indeed, for all valuations m of ∇ in M, we have that

$$\overline{\mathsf{e}}_{\overline{|\nabla|}}(t, \overline{m}) \ = \ F_{\nabla \vdash t}(M, [\mathsf{e}_n]_{n \in \mathbb{N}})(\iota, m) \ = \ F_{\nabla \vdash t'}(M, [\mathsf{e}_n]_{n \in \mathbb{N}})(\iota, m) \ = \ \overline{\mathsf{e}}_{\overline{|\nabla|}}(t', \overline{m})$$

where the first and last identities hold because the identity permutation extends ι.

To prove the only-if part, assume that (M, e) satisfies the judgement $\nabla \vdash t \approx t'$. Then, for $(\alpha, m) \in \Gamma_\nabla(M)$, as m is a valuation of $\alpha \cdot \nabla$ in M and (M, e) also satisfies $(\alpha \cdot \nabla) \vdash (\alpha \cdot t) \approx (\alpha \cdot t')$, we conclude that

$$
\begin{aligned}
F_{\nabla \vdash t}(M, [\mathsf{e}_n]_{n\in\mathbb{N}})(\alpha, m) &= \bar{\mathsf{e}}_{\overline{|\nabla|}}(\alpha \cdot t, \overline{m}) \\
&= \bar{\mathsf{e}}_{\overline{|\nabla|}}(\alpha \cdot t', \overline{m}) \\
&= F_{\nabla \vdash t'}(M, [\mathsf{e}_n]_{n\in\mathbb{N}})(\alpha, m)
\end{aligned}
\qquad \square
$$

Aiming at applying Theorems 6.1 and 6.2 we establish the following result.

Theorem 7.4 *The functorial signature and functorial context of the equational system associated to a NEL theory preserve filtered colimits and epimorphisms.*

PROOF. Let $\widetilde{\Sigma}$ and $\widetilde{\Gamma}$ respectively be the functorial signature and functorial context associated to a NEL theory $\mathbb{T} = (\Sigma, E)$.

As the product is closed, the functor $\Sigma(n) \times (-)^n$ preserves filtered colimits and epimorphisms for all $n \in \mathbb{N}$. Thus, the functorial signature $\widetilde{\Sigma}$, being the (pointwise) coproduct of these functors, also preserves filtered colimits and epimorphisms.

Since the functorial context $\widetilde{\Gamma}$ is the (pointwise) coproduct of functorial contexts of the form Γ_∇, it is enough to show that such functors preserve (i) filtered colimits and (ii) epimorphisms.

To show (i), we make the key observation that for all freshness contexts ∇, the following diagram is a pullback

$$
\begin{array}{ccc}
\Gamma_\nabla(M) & \xrightarrow{\;\jmath'_M\;} & \prod_{x\in|\nabla|}\left(\mathbb{A}^{\#(\nabla^{\#}(x))} \# M\right) \\
{\scriptstyle \imath'_M}\big\downarrow & & \big\downarrow{\scriptstyle \imath_M} \\
\mathbb{A}^{\#\nabla^{\mathcal{A}}} \times M^{|\nabla|} & \xrightarrow[\;\jmath_M\;]{} & \prod_{x\in|\nabla|}\left(\mathbb{A}^{\#(\nabla^{\#}(x))} \times M\right)
\end{array}
$$

where $\imath_M$ is induced by the embedding of the separating tensor into the product; $\imath'_M$ is the embedding determined by the definition of $\Gamma_\nabla(M)$; and $\jmath'_M$ is the restriction of the equivariant function $\jmath_M : (\alpha, m) \mapsto \Big((\alpha \restriction \nabla^{\#}(x), m(x))\Big)_{x\in|\nabla|}$, where $\alpha \restriction \nabla^{\#}(x)$ is the restriction of α to $\nabla^{\#}(x)$ (that is, the composite $\nabla^{\#}(x) \hookrightarrow \nabla^{\mathcal{A}} \xrightarrow{\alpha} \mathsf{A}$).

Thus, since the category of nominal sets is locally finitely presentable, and hence in it finite limits commute with filtered colimits, and since both the product and separating tensor are closed, and hence preserve filtered colimits, it follows that Γ_∇ preserves filtered colimits. Indeed, for D a filtered diagram of nominal sets, we have that

$$
\begin{aligned}
&\operatorname{colim}(\Gamma_\nabla D)\\
&\cong \operatorname{colim}\Big(\lim\big(\ \mathbb{A}^{\#\nabla_\mathcal{A}} \times D^{|\nabla|} \longrightarrow \textstyle\prod_{x\in|\nabla|} \mathbb{A}^{\#(\nabla^{\#}(x))} \times D \hookleftarrow \textstyle\prod_{x\in|\nabla|} \mathbb{A}^{\#(\nabla^{\#}(x))}\#D\ \big)\Big)\\
&\cong \lim\Big(\operatorname{colim}(\mathbb{A}^{\#\nabla_\mathcal{A}} \times D^{|\nabla|}) \longrightarrow \operatorname{colim}(\textstyle\prod_{x\in|\nabla|} \mathbb{A}^{\#(\nabla^{\#}(x))} \times D) \hookleftarrow \operatorname{colim}(\textstyle\prod_{x\in|\nabla|} \mathbb{A}^{\#(\nabla^{\#}(x))}\#D)\Big)\\
&\cong \lim\Big(\mathbb{A}^{\#\nabla_\mathcal{A}} \times (\operatorname{colim} D)^{|\nabla|} \longrightarrow \textstyle\prod_{x\in|\nabla|} \mathbb{A}^{\#(\nabla^{\#}(x))} \times (\operatorname{colim} D) \hookleftarrow \textstyle\prod_{x\in|\nabla|} \mathbb{A}^{\#(\nabla^{\#}(x))}\#(\operatorname{colim} D)\Big)\\
&\cong \Gamma_\nabla(\operatorname{colim} D)\ .
\end{aligned}
$$

To show (ii), we just need to show that Γ_∇ preserves surjectivity. To this end, let $f : P \twoheadrightarrow Q$ be a surjective equivariant function and let $(\alpha, q) \in \Gamma_\nabla(Q)$. Then, for every $x \in |\nabla|$, there exists $p_x \in P$ such that $f(p_x) = q(x)$. Moreover, since $\mathsf{supp}(p_x) \supseteq \mathsf{supp}(q(x))$ and $\mathsf{supp}(q(x)) \# (\alpha \cdot \nabla)^{\#}(x)$, there exists $\pi_x \in \mathfrak{S}_0(\mathsf{A})$ such that

$$\pi_x(a) = a \text{ for all } a \in \mathsf{supp}(q(x)) \quad\text{and}\quad \pi_x(\mathsf{supp}(p_x)) \# (\alpha \cdot \nabla)^{\#}(x)\ .$$

It follows that $f(\pi_x \cdot p_x) = \pi_x \cdot f(p_x) = \pi_x \cdot q(x) = q(x)$ and $(\alpha \cdot \nabla)^{\#}(x) \# \pi_x \cdot p_x$. Thus, setting $p'(x) = \pi_x \cdot p_x$ for all $x \in |\nabla|$, we have $(\alpha, p') \in \Gamma_\nabla(P)$ with $\Gamma_\nabla(f)(\alpha, p') = (\alpha, fp') = (\alpha, q)$ as required. □

Corollary 7.5 *The category of algebras for a NEL theory is cocomplete and monadic over nominal sets, with the induced free-algebra monad being finitary and epicontinuous. Moreover, free algebras on nominal sets are constructed in $\omega + \omega$ steps by the construction (2) followed by the construction (4) in Section 4.*

7.3.4 Presentation of free algebras

We proceed to give an inductive presentation of free algebras for NEL theories.

For a NEL theory $\mathbb{T} = (\Sigma, E)$ and its associated equational system $\widetilde{\mathbb{T}}$, we have the following situation:

$$
\begin{array}{ccc}
\mathbb{T}\text{-}\mathbf{Alg} \cong \widetilde{\mathbb{T}}\text{-}\mathbf{Alg} & \underset{\hookrightarrow}{\overset{\longleftarrow}{\bot}} & \widetilde{\Sigma}\text{-}\mathbf{Alg} \cong \Sigma\text{-}\mathbf{Alg}\\
 & & \uparrow \dashv \downarrow\\
 & & \mathbf{Nom}
\end{array}
$$

$$\mathsf{Ref}\ \frac{m \in M}{m \approx_E m} \qquad \mathsf{Sym}\ \frac{m \approx_E m'}{m' \approx_E m} \qquad \mathsf{Trans}\ \frac{m \approx_E m' \quad m' \approx_E m''}{m \approx_E m''}$$

$$\mathsf{Axiom}\ \frac{(\alpha, m) \in \Gamma_\nabla(M)}{\overline{\mathsf{e}}_{\overline{|\nabla|}}(\alpha \cdot t, \overline{m}) \ \approx_E \ \overline{\mathsf{e}}_{\overline{|\nabla|}}(\alpha \cdot t', \overline{m})} \ \Big((\nabla \vdash t \approx t') \in E\Big)$$

$$\mathsf{Cong}\ \frac{m_i \approx_E m_i' \ \ (1 \le i \le k)}{\mathsf{o}\ m_1 \ldots m_k \approx_E \mathsf{o}\ m_1' \ldots m_k'} \ \Big(\mathsf{o} \in \Sigma(k)\Big)$$

Fig. 1. Rules for the relation $\approx_E$.

By the construction (2), the free $\widetilde{\Sigma}$-algebra on a nominal set V has as carrier the nominal set $T_\Sigma(V)$ inductively defined by the rules (6).

We obtain a presentation of the free $\widetilde{\mathbb{T}}$-algebra on a $\widetilde{\Sigma}$-algebra $(M, [\mathsf{e}_n]_{n\in\mathbb{N}})$ by analyzing the construction (4). Since the forgetful functor $|-| : \mathbf{Nom} \to \mathbf{Set}$ creates colimits, it follows from the standard construction of colimits in **Set** that the underlying set of the carrier object of the free $\widetilde{\mathbb{T}}$-algebra on $(M, [\mathsf{e}_n]_{n\in\mathbb{N}})$ is obtained as the colimit of the ω-chain of quotients

$$|M| \twoheadrightarrow |M|/_{\approx_1} \twoheadrightarrow \cdots \twoheadrightarrow |M|/_{\approx_n} \twoheadrightarrow \cdots$$

where $\approx_n$ denotes the equivalence relation on $|M|$ generated by the following rules

$$\text{for } \approx_1 \ : \quad \frac{(\alpha, m) \in \Gamma_\nabla(M)}{\overline{\mathsf{e}}_{\overline{|\nabla|}}(\alpha \cdot t, \overline{m}) \approx_1 \overline{\mathsf{e}}_{\overline{|\nabla|}}(\alpha \cdot t', \overline{m})} \ \Big((\nabla \vdash t \approx t') \in E\Big)$$

$$\text{for } \approx_n \ (n \ge 2) \ : \quad \frac{m \approx_{n-1} m'}{m \approx_n m'} \qquad \frac{m_i \approx_{n-1} m_i' \ \ (1 \le i \le k)}{\mathsf{o}\ m_1 \ldots m_k \approx_n \mathsf{o}\ m_1' \ldots m_k'} \ (\mathsf{o} \in \Sigma(k))$$

Thus, the free $\widetilde{\mathbb{T}}$-algebra on $(M, [\mathsf{e}_n]_{n\in\mathbb{N}})$ has carrier object $M/_{\approx_E}$ given by the underlying set $|M|/_{\approx_E}$ for $\approx_E$ the equivalence relation on $|M|$ given by the rules in Figure 1 together with the action given by $\pi \cdot [m]_{\approx_E} = [\pi \cdot m]_{\approx_E}$. Furthermore, the quotient map $M \twoheadrightarrow M/_{\approx_E}$ sends $m \in M$ to $[m]_{\approx_E}$, and the $\widetilde{\Sigma}$-algebra structure $\Big[[\mathsf{e}_n]_{\approx_E}\Big]_{n\in\mathbb{N}}$ on $M/_{\approx_E}$ is given by

$$[\mathsf{e}_n]_{\approx_E}(\mathsf{o}, [m_1]_{\approx_E}, \ldots, [m_n]_{\approx_E}) \ = \ [\mathsf{e}_n(\mathsf{o}, m_1, \ldots, m_n)]_{\approx_E}$$

for all $n \in \mathbb{N}$, $\mathsf{o} \in \Sigma(n)$, and $m_1, \ldots, m_n \in M$.

As a corollary, we now establish the following ground completeness result for NEL [7, Theorem 9.4]:

> For a NEL theory $\mathbb{T}$, if a ground judgement (*viz.*, a judgement with no variables) is satisfied by all $\mathbb{T}$-algebras, then the judgement is provable in NEL.

The free $\mathbb{T}$-algebra on the empty nominal set $\emptyset$ consists of the nominal set $T_\Sigma(\emptyset)/_{\approx_E}$ of ground terms $T_\Sigma(\emptyset)$ quotiented by $\approx_E$, equipped with the syntactic Σ-structure e. Thus, for any ground judgement $(a_1, \ldots, a_m \blacktriangleright \{\ \} \vdash t \equiv t')$, we have that

$$\begin{aligned}
&\text{every } \mathbb{T}\text{-algebra satisfies } (a_1, \ldots, a_m \blacktriangleright \{\ \} \vdash t \equiv t') \\
&\quad\Longrightarrow\ (T_\Sigma(\emptyset)/_{\approx_E}, \mathsf{e}) \text{ satisfies } (a_1, \ldots, a_m \blacktriangleright \{\ \} \vdash t \equiv t') \\
&\quad\Longrightarrow\ \bar{\mathsf{e}}_{\overline{\emptyset}}(\, t \,,\, \overline{()}\,) \;=\; \bar{\mathsf{e}}_{\overline{\emptyset}}(\, t' \,,\, \overline{()}\,) \quad \text{in } T_\Sigma(\emptyset)/_{\approx_E} \\
&\quad\Longrightarrow\ [t]_{\approx_E} \;=\; [t']_{\approx_E} \quad \text{in } T_\Sigma(\emptyset)/_{\approx_E} \\
&\quad\Longrightarrow\ t \approx_E t' \ \text{ is derivable from the rules in Figure 1}
\end{aligned}$$

The ground completeness result follows by noticing that every proof of $t \approx_E t'$ is easily turned into a proof of the judgement $(a_1, \ldots, a_m \blacktriangleright \{\ \} \vdash t \equiv t')$ in NEL.

8 Conclusion

The main and salient contribution of this paper can be summarised as the introduction of a framework for the specification of equational systems (Section 3) and the development of an associated theory of free constructions (Sections 4 and 5) that is general (see Sections 3.4 and 6.4) and, most importantly, practical as needed in modern applications (see Section 7).

In connection to related work, we have learnt during the course of this work that variations on the concept of equational system, and its dual of equational cosystem (*viz.*, an equational system on an opposite base category), had already been considered in the literature. For instance, Fokkinga [14] introduces the more general concept of law between the so-called transformers, but only studies initial algebras for the laws that are equational systems; Cîrstea [6] introduces the concept of coequation between abstract cosignatures, which is equivalent to our notion of equational cosystem, and studies final coalgebras for them; Ghani, Lüth, De Marchi, and Power [18] introduce the concept of functorial coequational presentations, which is equivalent to our notion of equational cosystem on a locally presentable base category with an accessible functorial signature and an accessible functorial context, and study cofree constructions for them.

In comparison, our theory of equational (co)systems is more general and comprehensive than that of [14] and [6], and it can be related to that of [18] as follows. The proof of the dual of Corollary 5.9 (3) together with the construction of cofree coalgebras for endofunctors by terminal sequences of Worrell [33], gives a construction of cofree coalgebras for equational cosystems on a locally presentable base category with an accessible functorial signature that preserves monomorphisms. This is a variation of a main result of the theory developed by Ghani, Lüth, De Marchi, and Power [18] (see *e.g.* their Lemmas 5.8 and 5.14); which is there proved by means of the theory of accessible categories without assuming the preservation of monomorphisms but assuming an accessible arity endofunctor.

In the context of the enriched algebraic theories of Kelly and Power [23], which we have exhibited as equational systems in Section 3.4 (2), one may also consider the categorical presentation of term rewriting via coinserters of Ghani and Lüth [17] in the setting of algebraic theories on the category of preorders. In this vein, we have developed a theory of free constructions for *inequational systems* in an abstract-rewriting enriched setting together with a logical theory for rewriting modulo equations. Details will appear elsewhere.

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