# Categorical Equational System

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→ Part I. Equational System

- Signatures
- Equations
- Models (Algebras)
- Construction of free algebras

Part II. Term Equational System

- Signatures
- Equations
- Models (Algebras)
- Equational Reasoning

# Algebraic Theories

e.g. the theory $\mathbb{G}$ of groups

- signature

$$\Sigma = \{ e:0 \quad i:1 \quad m:2 \}$$

- equation

$$x \vdash m(x,e) = x$$
$$x \vdash m(i(x),x) = e$$
$$x,y,z \vdash m(m(x,y),z) = m(x,m(y,z))$$

- Algebras

$X \in \underline{Set}$ with $[e]: 1 \to X \quad [i]: X \to X \quad [m]: X^2 \to X$

s.t.

$$\forall x \in X \quad [m](x,[e]) = [x]$$
$$\forall x \in X \quad [m]([i](x),x) = [e]$$
$$\forall x,y,z \in X \quad [m]([m](x,y),z) = [m](x,[m](y,z))$$

# Free construction for Algebraic Theories

- Free Algebras

$$\begin{tikzcd} (T_{\mathcal{G}}X,[e],[i],[m]) & \mathcal{G}\text{-Alg} \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & (X,[e],[i],[m]) \arrow[d, mapsto] \\ X \arrow[u, mapsto] & \mathrm{Set} \arrow[u, bend left] & X \end{tikzcd}$$

- Free Construction

$$\begin{tikzcd} (T_{\mathcal{G}}X,[e],[i],[m]) & & (T_{\Sigma}X,[e],[i],[m]) \arrow[ll, mapsto] \\ \mathcal{G}\text{-Alg} \arrow[drr] & \Sigma\text{-Alg} \arrow[l, bend right] \arrow[l, phantom, "\perp"] \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & \\ & \mathrm{Set} \arrow[u, bend left] & X \arrow[uu, mapsto] \end{tikzcd}$$

X with [e],[i][m] satisfying the group axioms

X with [e],[i],[m]

(1) $T_\Sigma X \ni t ::= x \in X \mid e \mid i(t) \mid m(t_1,t_2)$

(2) $T_{\mathcal{G}}X = T_\Sigma X/\approx$

$$\frac{}{e \approx e} \qquad \frac{x \approx x'}{i(x) \approx i(x')} \qquad \frac{x \approx x' \quad y \approx y'}{m(x,y) \approx m(x',y')}$$

$$\frac{x \in X}{m(x,e) \approx x} \qquad \frac{x \in X}{m(x,i(x)) \approx e} \qquad \frac{x,y,z \in X}{m(m(x,y),z) \approx m(x,m(y,z))}$$

# Motivation for Equational System

- Non-classical signatures and equations
  - Pi-algebras (Stark)

    $X \in \mathrm{Set}^{\mathbb{I}}$ with $[\mathrm{in}] : N \times X^N \to X$, $[\mathrm{new}] : (N \multimap X) \to X$, ...

    satisfying

    $p : N \multimap X^N \vdash \mathrm{new}(\nu x{:}N.\ \mathrm{in}(x, p@x)) = \mathrm{nil} : X$

    $\vdots$
  - $\Sigma$-Algebras (Fiore, Plotkin & Turi)

    $X \in \mathrm{Set}^{\mathbb{F}}$ with $\Sigma X \to X$, $X \bullet X \to X$, ...

    satisfying ...
  - Nominal agebras (Gabbay & Mathijssen ; Pitts & Clouston)

    $X \in \mathrm{Nom}$ with $[A]X \to X$, $X^2 \to X$, ...

    satisfying ...

# Equational System

- Signature

$\Sigma = \{e:0,\ i:1,\ m:2\}$ $\rightsquigarrow$ $\circlearrowright^{\Sigma X = 1 + X + X^2}$ Set $\rightsquigarrow$ $\circlearrowright^{\Sigma}$ $\mathcal{C}$

$X$ with $[e], [i], [m]$ $\qquad$ $X$ with $[[e],[i],[m]] : \Sigma X \to X$ $\qquad$ $(X,\ s : \Sigma X \to X)$

- Equation

$X, [e], [i], [m]$ satisfies $x \vdash m(i(x), x) = e,\quad x \vdash \cdots,\quad x, y, z \vdash \cdots$

$\Updownarrow$

$x, [e], [i], [m]$ satisfies $X \xrightarrow{\langle [i], id_X \rangle} X^2 \xrightarrow{[m]} X$, $\ X \xrightarrow{!} 1 \xrightarrow{[e]} X$, $\quad X \underset{\cdots}{\overset{\cdots}{\rightrightarrows}} X, \quad X^3 \underset{\cdots}{\overset{\cdots}{\rightrightarrows}} X$

$\Updownarrow$

$(X, [[e],[i],[m]] : 1 + X + X^2 \to X)$ satisfies $X + X + X^3 \underset{\cdots}{\overset{\cdots}{\rightrightarrows}} X$

$\Updownarrow$

$(X,\ s : \Sigma X \to X)$ satisfies $\Gamma X \underset{R(X,s)}{\overset{L(X,s)}{\rightrightarrows}} X$

$\hookrightarrow \left( \Sigma\text{-Alg} \underset{R}{\overset{L}{\rightrightarrows}} \Gamma\text{-Alg} \right)$

# Equational System

$$\boxed{(X,\ s : \Sigma X \to X) \quad \text{satisfies} \quad \Gamma X \underset{R(X,s)}{\overset{L(X,s)}{\rightrightarrows}} X}$$

$$\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$$

where

$$\begin{tikzcd}
\mathbb{S}\text{-Alg} \arrow[r, hook] \arrow[dr] & \Sigma\text{-Alg} \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] \arrow[d] & \Gamma\text{-Alg} \arrow[dl] \\
& \mathscr{C} &
\end{tikzcd}$$

$(X, s : \Sigma X \to X)$
satisfying
$L(X,s) = R(X,s)$

# Properties

$$
\begin{tikzcd}
\mathbb{S}\text{-Alg} \arrow[r, hook, "I"] \arrow[dr] & \Sigma\text{-Alg} \arrow[l, bend right, "K_{\mathbb{S}}"'] \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] \arrow[d, shift left] & \Gamma\text{-Alg} \\
& \mathcal{C} \arrow[u, shift left, "F_\Sigma"] \arrow[u, phantom, "\dashv"] &
\end{tikzcd}
$$

- If $\mathcal{C}$ is cocomplete,
  $\Sigma, \Gamma$ preserve colimits of $\omega$-chains

  then

  1. $F_\Sigma, K_{\mathbb{S}}$ exists
  2. $\mathbb{S}$-Alg is cocomplete
  3. $\mathbb{S}$-Alg is monadic over $\mathcal{C}$.

# Explicit free construction

$$
\begin{tikzcd}
(T_{\mathcal{S}}X, \tau^{\mathcal{S}}_X : \Sigma T_{\mathcal{S}}X \to T_{\mathcal{S}}X) & & (T_\Sigma X, \tau^\Sigma_X : \Sigma T_\Sigma X \to T_\Sigma X) \arrow[ll] \\
\mathcal{S}\text{-Alg} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \Sigma\text{-Alg} \arrow[l, shift right, "K_{\mathcal{S}}"'] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \\
& \mathcal{C} \arrow[u, shift left, "F_\Sigma"] & X \arrow[uu]
\end{tikzcd}
$$

- Construction of $F_\Sigma$

$$
\begin{aligned}
T_\Sigma X \ni t ::=\ & x \in X \\
\mid\ & e \\
\mid\ & i(t) \\
\mid\ & m(t_1, t_2)
\end{aligned}
\quad \rightsquigarrow \quad
\emptyset \to X + \Sigma\emptyset \to X + \Sigma(X + \Sigma\emptyset) \to \cdots \to T_\Sigma X
$$

- Construction of $K_{\mathcal{S}}$ (additionally assume $\Sigma, \Gamma$ preserve epimorphisms)

$T_{\mathcal{S}}X = T_\Sigma X/\approx$

$$
\cdots \frac{}{m(i(x),x) \approx e}(x \in X) \cdots
$$

$$
\cdots \frac{x \approx x' \quad y \approx y'}{m(x,y) \approx m(x',y')} \cdots
$$

$\rightsquigarrow$

$$
\begin{tikzcd}
& \Sigma(T_\Sigma X) \arrow[r, two heads] \arrow[d, "\tau^\Sigma_X"'] & \Sigma(T_\Sigma X)_1 \arrow[r, two heads] \arrow[dr] & \Sigma(T_\Sigma X)_2 \arrow[r, two heads] \arrow[dr] & \cdots & \Sigma(T_{\mathcal{S}}X) \arrow[d, "\tau^{\mathcal{S}}_X"] \\
\Gamma(T_\Sigma X) \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] & T_\Sigma X \arrow[r, two heads, "\text{coeq}"'] \arrow[ur, phantom, "\text{P.O.}"] & (T_\Sigma X)_1 \arrow[r, two heads] \arrow[ur, phantom, "\text{P.O.}"] & (T_\Sigma X)_2 \arrow[r, two heads] & (T_\Sigma X)_3 \cdots & T_{\mathcal{S}}X
\end{tikzcd}
$$

axiom rules

congruence rules

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## → Part II. Term Equational System

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# Term Equational System (TES)

- Signature

$$\Sigma \circlearrowright \mathcal{C}$$

(1) $\mathcal{C}, \otimes, [-,=]$ symmetric monoidal closed category.

(2) $\Sigma$ has a strength $st : X \otimes \Sigma Y \to \Sigma(X \otimes Y)$.

(3) $\Sigma\text{-Alg} \underset{\mathcal{C}}{\overset{\uparrow \dashv \downarrow}{}}$ $\quad \left( T_\Sigma X,\ \tau^\Sigma_X : \Sigma T_\Sigma X \to T_\Sigma X \right)$ $\quad \uparrow X$

- Equation

$x, y, z \vdash m(m(x,y),z) \equiv m(x, m(y,z))$

$\leadsto$

$m(m(x,y),z),\ m(x, m(y,z)) \in T_\Sigma\{x,y,z\}$ $\quad$ where $t \in T_\Sigma X := x \in X \mid e \mid i(t) \mid m(t_1, t_2)$

$\leadsto$

$m(m(x,y),z),\ m(x, m(y,z)) : 1 \rightrightarrows T_\Sigma\{x,y,z\}$

$\leadsto$

$l, r : 1 \rightrightarrows T_\Sigma C$ for $C \in \mathcal{C}$

$\leadsto$

$l, r : A \rightrightarrows T_\Sigma C$ for $A, C \in \mathcal{C}$

# Model Theory

For $\mathcal{S} = (\mathcal{C}, \Sigma, E)$

An algebra for $\mathcal{S}$ is

$$(X, s : \Sigma X \to X)$$

satisfying

every $(l, r : A \to T_\Sigma C) \in E$.

$$[l], [r] : \Sigma\text{-Alg} \rightrightarrows ([C,-] \otimes A)\text{-Alg}$$

Technically

$[l](X, s : \Sigma X \to X)$

$= [C,X] \otimes A \to [C,X] \otimes TC$

$\to T([C,X] \otimes C) \to TX \to X$

e.g. $m(m(x,y),z), m(x,m(y,z)) \in 1 \rightrightarrows T_\Sigma\{x,y,z\}$

$$[m(m(x,y),z)] : (X, [e], [i], [m]) \longrightarrow (X, X^{\{x,y,z\}} \times 1 \longrightarrow X)$$

# Term Equational Logic (TEL)

- For a TES $\mathcal{C} \circlearrowleft \Sigma$ with $E$ where $U, V \vdash t \equiv t' \triangleq t \equiv t' : U \to T_\Sigma V$

  - Equivalence relation

$$\frac{}{U, V \vdash t \equiv t} \qquad \frac{U, V \vdash t \equiv t'}{U, V \vdash t' \equiv t} \qquad \frac{U, V \vdash t \equiv t' \quad U, V \vdash t' \equiv t''}{U, V \vdash t \equiv t''}$$

  - Axiom

$$\frac{}{U, V \vdash t \equiv t'} \ (U, V \vdash t \equiv t') \in E$$

  - Substitution

$$\frac{U, W \vdash t \equiv t' \quad W, V \vdash s \equiv s'}{U, V \vdash t[s] \equiv t'[s']}$$

  - Tensor extension

$$\frac{U, V \vdash t \equiv t'}{W \otimes U, W \otimes V \vdash \langle W \rangle t \equiv \langle W \rangle t'}$$

  - Local character

$$\frac{U_i, V \vdash t \circ e_i \equiv t' \circ e_i \quad (i \in I)}{U, V \vdash t \equiv t'} \quad \{e_i : U_i \to U\}_{i \in I} \text{ jointly epi}$$

- Soundness

$U, V \vdash t \equiv t'$ is provable from $E$ using TEL

$\Longrightarrow$ $U, V \vdash t \equiv t'$ is satisfied by all models of $\mathcal{C} \circlearrowleft \Sigma_{\text{with } E}$

# Internal Completeness

For $\mathbb{S} = \mathcal{C}^{\partial\Sigma}$ with $E$

$U, V \vdash t \equiv t'$ is satisfied by all models of $\mathbb{S}$

$\Leftrightarrow$ $U, V \vdash t \equiv t'$ is satisfied by the free $\mathbb{S}$-algebra on $V$

$\left(T_{\mathbb{S}}V,\ \tau^{\mathbb{S}}_V : \Sigma(T_{\mathbb{S}}V) \to (T_{\mathbb{S}}V)\right)$

$\Leftrightarrow$ $U \underset{t'}{\overset{t}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\mathbb{S}} V$

under the assumption that

$$\begin{tikzcd} \mathbb{S}\text{-Alg} \arrow[r, hook] \arrow[r, phantom, shift right=2, "\perp"] & \Sigma\text{-Alg} \arrow[l, bend right] \end{tikzcd}$$

exists

# Towards complete rewriting system for TES

$$
\begin{tikzcd}
 & & \Sigma T_\Sigma V \arrow[r, two heads] \arrow[d, "\tau^\Sigma_V"'] \arrow[dr] & \Sigma(T_\Sigma V)_1 \arrow[r, two heads] \arrow[dr] \arrow[d, phantom, "\text{p.o.}"] & \Sigma(T_\Sigma V)_2 & \cdots & \Sigma(T_S V) \arrow[d, "\tau^S_V"] \\
{[C,TV]\otimes A} \arrow[rr, shift left] \arrow[rr, shift right] & & T_\Sigma V \arrow[r, two heads, "\text{coeq}"] \arrow[rrrr, bend right, "q_V"'] & (T_\Sigma V)_1 \arrow[r, two heads] & (T_\Sigma V)_2 & \cdots & T_S V
\end{tikzcd}
$$

The set of axioms $E$

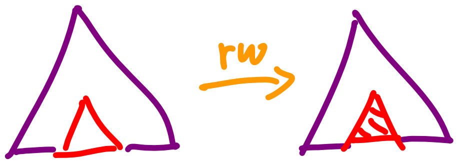
e.g. Theory of groups $G$  $(T_G V = T_\Sigma V/_\equiv)$

$$\frac{t \in T_\Sigma V}{m(t,e) \equiv t} \qquad \frac{t \in T_\Sigma V}{m(i(t),t) \equiv e} \qquad \frac{t_1,t_2,t_3 \in T_\Sigma V}{m(m(t_1,t_2),t_3) \equiv m(t_1, m(t_2,t_3))}$$

$$\frac{}{e \equiv e} \qquad \frac{t \equiv s}{i(t) \equiv i(s)} \qquad \frac{t_1 \equiv s_1 \quad t_2 \equiv s_2}{m(t_1,t_2) \equiv m(s_1,s_2)}$$

$$\boxed{1 \overset{l}{\underset{r}{\rightrightarrows}} T_\Sigma V \longrightarrow T_\Sigma V/_\equiv \iff [l]_\equiv = [r]_\equiv \text{ in } T_\Sigma V/_\equiv \iff l \equiv r \text{ is derivable}}$$

$\equiv$ in $T_\Sigma V \rightsquigarrow$ Bidirectional rewriting in $T_\Sigma V$



# Concluding Remarks

- Applications
  - Algebraic theories (Set)
  - Binding Term Equational Logic (Hamana) ($\mathrm{Set}^{\mathbb{I}}$)
  - Nominal Equational Logic (Gabbay & Mathijssen, Clouston & Pitts) (Nom)
  - Cominatory Reduction System (Klop) ($\mathrm{Set}^{\mathbb{F}}$)
    Second-order abstract syntax (Fiore)

- Further work
  - Abstract condition for confluence
  - Abstract condition for termination