Categorical Equational System

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→ Part I. Equational System

- Signatures
- Equations
- Models (Algebras)
- Construction of free algebras

Part II. Term Equational System

- Signatures
- Equations
- Models (Algebras)
- Equational Reasoning

Algebraic Theories

e.g. the theory $\mathbb{G}$ of groups

- signature

$$\Sigma = \{ e:0 \quad i:1 \quad m:2 \}$$

- equation

$$x \vdash m(x,e) = x$$
$$x \vdash m(i(x),x) = e$$
$$x,y,z \vdash m(m(x,y),z) = m(x,m(y,z))$$

- Algebras

$X \in \underline{\mathbf{Set}}$ with $[e]: 1 \to X \quad [i]: X \to X \quad [m]: X^2 \to X$

s.t.

$$\forall x \in X \quad [m](x,[e]) = [x]$$
$$\forall x \in X \quad [m]([i](x),x) = [e]$$
$$\forall x,y,z \in X \quad [m]([m](x,y),z) = [m](x,[m](y,z))$$

Free construction for Algebraic Theories

- Free Algebras

$$\begin{tikzcd}
(T_{\mathbb{G}}X, [e], [i], [m]) & \mathbb{G}\text{-Alg} \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & (X, [e], [i], [m]) \arrow[d, mapsto] \\
X \arrow[u, mapsto] & \mathrm{Set} \arrow[u, bend left] & X
\end{tikzcd}$$

- Free Construction

$$\begin{tikzcd}
(T_{\mathbb{G}}X, [e], [i], [m]) & & (T_{\Sigma}X, [e], [i], [m]) \arrow[ll, mapsto] \\
\mathbb{G}\text{-Alg} \arrow[r, bend left, phantom, "\perp"] \arrow[dr] & \Sigma\text{-Alg} \arrow[l, bend right] \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & \\
& \mathrm{Set} \arrow[u, bend left] & X \arrow[uu, mapsto]
\end{tikzcd}$$

$\mathbb{G}$-Alg: X with $[e], [i] [m]$ satisfying the group axioms

Σ-Alg: X with $[e], [i], [m]$

(1) $T_\Sigma X \ni t ::= x \in X$
$\quad | \ e$
$\quad | \ i(t)$
$\quad | \ m(t_1, t_2)$

(2) $T_{\mathbb{G}}X = T_\Sigma X / \approx$

$$\frac{}{e \approx e} \qquad \frac{x \approx x'}{i(x) \approx i(x')} \qquad \frac{x \approx x' \quad y \approx y'}{m(x,y) \approx m(x', y')}$$

$$\frac{x \in X}{m(x,e) \approx x} \qquad \frac{x \in X}{m(x, i(x)) \approx e} \qquad \frac{x, y, z \in X}{m(m(x,y), z) \approx m(x, m(y,z))}$$

Motivation for Equational System

- Non-classical signatures and equations
  - Pi-algebras (Stark)

    $X \in \mathrm{Set}^{\mathbb{I}}$ with $[\mathrm{in}] : N \times X^N \to X$, $[\mathrm{new}] : (N \multimap X) \to X$, ...

    satisfying

    $p : N \multimap X^N \vdash \mathrm{new}(\nu x{:}N.\ \mathrm{in}(x, p@x)) = \mathrm{nil} : X$

    $\vdots$
  - Σ-Algebras (Fiore, Plotkin & Turi)

    $X \in \mathrm{Set}^{\mathbb{F}}$ with $\Sigma X \to X$, $X \bullet X \to X$, ...

    satisfying ...

  - Nominal agebras (Gabbay & Mathijssen ; Pitts & Clouston)

    $X \in \mathrm{Nom}$ with $[\mathbb{A}]X \to X$, $X^2 \to X$, ...

    satisfying ...

Equational System

- Signature

$\Sigma = \{e:0, i:1, m:2\}$ $\rightsquigarrow$ $\circlearrowleft^{\Sigma X = 1+X+X^2}$ Set $\rightsquigarrow$ $\circlearrowleft^{\Sigma}$ $\mathcal{C}$

X with $[e],[i],[m]$ $\quad$ X with $[[e],[i],[m]] : \Sigma X \to X$ $\quad$ $(X, s : \Sigma X \to X)$

- Equation

$X, [e], [i], [m]$ satisfies $x \vdash m(i(x), x) = e$, $x \vdash \cdots$, $x, y, z \vdash \cdots$

$\Updownarrow$

$X, [e], [i], [m]$ satisfies

$$\begin{tikzcd} & X^2 \arrow[dr, "{[m]}"] & \\ X \arrow[ur, "{\langle [i], id_X \rangle}"] \arrow[dr, "!"'] & & X \\ & 1 \arrow[ur, "{[e]}"'] & \end{tikzcd}, \quad X \underset{\cdots}{\overset{\cdots}{\rightrightarrows}} X, \quad X^3 \underset{\cdots}{\overset{\cdots}{\rightrightarrows}} X$$

$\Updownarrow$

$(X, [[e],[i],[m]] : 1+X+X^2 \to X)$ satisfies $X + X + X^3 \underset{\cdots}{\overset{\cdots}{\rightrightarrows}} X$

$\Updownarrow$

$(X, s : \Sigma X \to X)$ satisfies $\Gamma X \underset{R(X,s)}{\overset{L(X,s)}{\rightrightarrows}} X$

$\hookrightarrow$ $\left(\Sigma\text{-Alg} \underset{R}{\overset{L}{\rightrightarrows}} \Gamma\text{-Alg} \right)$

Equational System

$$\boxed{(X,\ s : \Sigma X \to X) \quad \text{satisfies} \quad \Gamma X \overset{L(X,s)}{\underset{R(X,s)}{\rightrightarrows}} X}$$

$$\mathcal{S} = (\mathcal{C} : \Sigma \rhd \Gamma \vdash L = R)$$

where

$$\begin{tikzcd}
\mathcal{S}\text{-Alg} \arrow[r, hook] \arrow[dr] \arrow[d, squiggly] & \Sigma\text{-Alg} \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] \arrow[d] & \Gamma\text{-Alg} \arrow[dl] \\
\begin{array}{l}(X, s : \Sigma X \to X) \\ \text{satisfying} \\ \quad L(X,s) = R(X,s)\end{array} & \mathcal{C} &
\end{tikzcd}$$

Properties

$$
\begin{tikzcd}
\mathbb{S}\text{-Alg} \arrow[r, hook, "I"] \arrow[dr] & \Sigma\text{-Alg} \arrow[l, bend right, "K_{\mathbb{S}}"'] \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] \arrow[d, shift left] & \Gamma\text{-Alg} \\
& \mathcal{C} \arrow[u, shift left, "F_\Sigma"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

- If $\mathcal{C}$ is cocomplete,
  Σ, Γ preserve colimits of ω-chains
  then

  ① $F_\Sigma, K_{\mathbb{S}}$ exists

  ② $\mathbb{S}$-Alg is cocomplete

  ③ $\mathbb{S}$-Alg is monadic over $\mathcal{C}$.

Explicit free construction

$$
\begin{tikzcd}
(T_{\mathcal{S}}X,\ \tau^{\mathcal{S}}_X : \Sigma T_{\mathcal{S}}X \to T_{\mathcal{S}}X) & & (T_\Sigma X,\ \tau^{\Sigma}_X : \Sigma T_\Sigma X \to T_\Sigma X) \arrow[ll] \\
\mathcal{S}\text{-Alg} \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \Sigma\text{-Alg} \arrow[l, bend right, "K_{\mathcal{S}}"'] \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & \\
 & \mathcal{C} \arrow[u, bend left, "F_\Sigma"] & X \arrow[uu]
\end{tikzcd}
$$

- Construction of F_Σ

$$
\begin{aligned}
T_\Sigma X \ni t ::=\ & x \in X \\
|\ & e \\
|\ & i(t) \\
|\ & m(t_1, t_2)
\end{aligned}
\quad \rightsquigarrow \quad
\emptyset \longrightarrow X+\Sigma\emptyset \longrightarrow X+\Sigma(X+\Sigma\emptyset) \to \cdots \longrightarrow T_\Sigma X
$$

- Construction of $K_{\mathcal{S}}$ (additionally assume Σ, Γ preserve epimorphisms)

$T_{\mathcal{S}}X = T_\Sigma X/_{\approx}$

$$
\cdots \quad \frac{}{m(i(x),x) \approx e}\,(x \in X) \quad \cdots
$$

$$
\cdots \quad \frac{x \approx x' \quad y \approx y'}{m(x,y) \approx m(x',y')} \quad \cdots
$$

$\rightsquigarrow$

$$
\begin{tikzcd}
 & \Sigma(T_\Sigma X) \arrow[r, two heads] \arrow[d, "\tau^\Sigma_X"'] \arrow[dr, phantom, "\text{P.O.}"] & \Sigma(T_\Sigma X)_1 \arrow[r, two heads] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\text{P.O.}"] & \Sigma(T_\Sigma X)_2 \arrow[r, two heads] \arrow[dr] & \cdots & \Sigma(T_{\mathcal{S}}X) \arrow[d, "\tau^{\mathcal{S}}_X"] \\
\Gamma(T_\Sigma X) \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] & T_\Sigma X \arrow[r, two heads, "\text{coeq}"'] & (T_\Sigma X)_1 \arrow[r, two heads] & (T_\Sigma X)_2 \arrow[r, two heads] & (T_\Sigma X)_3 \quad \cdots & T_{\mathcal{S}}X
\end{tikzcd}
$$

axiom rules

congruence rules

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Term Equational System (TES)

- Signature

$$\begin{tikzcd} \mathscr{C} \arrow[loop above, "\Sigma"] \end{tikzcd}$$

(1) $\mathscr{C}, \otimes, [-,=]$ symmetric monoidal closed category.

(2) Σ has a strength $st: X \otimes \Sigma Y \to \Sigma(X \otimes Y)$.

(3) $\Sigma\text{-Alg} \quad (T_\Sigma X,\ \tau^\Sigma_X : \Sigma T_\Sigma X \to T_\Sigma X)$

$$\begin{tikzcd} \Sigma\text{-Alg} \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\ \mathscr{C} \arrow[u, bend left] \end{tikzcd} \qquad \begin{tikzcd} (T_\Sigma X, \tau^\Sigma_X) \\ X \arrow[u, mapsto] \end{tikzcd}$$

- Equation

$x, y, z \vdash m(m(x,y),z) \equiv m(x, m(y,z))$

$\rightsquigarrow$

$m(m(x,y),z),\ m(x,m(y,z)) \in T_\Sigma\{x,y,z\}$ where $t \in T_\Sigma X := x \in X \mid e \mid i(t) \mid m(t_1,t_2)$

$\rightsquigarrow$

$m(m(x,y),z),\ m(x,m(y,z)) : 1 \rightrightarrows T_\Sigma\{x,y,z\}$

$\rightsquigarrow$

$l, r : 1 \rightrightarrows T_\Sigma C$ for $C \in \mathscr{C}$

$\rightsquigarrow$

$l, r : A \rightrightarrows T_\Sigma C$ for $A, C \in \mathscr{C}$

Model Theory

For $\mathbb{S} = (\mathcal{C}, \Sigma, E)$

An algebra for $\mathbb{S}$ is

$$(X, s : \Sigma X \to X)$$

satisfying

every $(\ell, r : A \to T_\Sigma C) \in E$.

Technically

$[\ell](X, s : \Sigma X \to X)$

$= [C, X] \otimes A \to [C, X] \otimes TC$

$\to T([C, X] \otimes C) \to TX \to X$

$$[\ell], [r] : \Sigma\text{-Alg} \rightrightarrows ([C, -] \otimes A)\text{-Alg}$$

e.g. $m(m(x,y),z), m(x, m(y,z)) \in 1 \rightrightarrows T_\Sigma\{x,y,z\}$

$$[m(m(x,y),z)] : (X, [e], [i], [m]) \longrightarrow (X, X^{\{x,y,z\}} \times 1 \longrightarrow X)$$

Term Equational Logic (TEL)

- For a TES $\mathcal{C} \circlearrowleft \Sigma$ with E where $U,V \vdash t \equiv t' \triangleq t \equiv t' : U \to T_\Sigma V$

  - Equivalence relation

$$\frac{}{U,V \vdash t \equiv t} \qquad \frac{U,V \vdash t \equiv t'}{U,V \vdash t' \equiv t} \qquad \frac{U,V \vdash t \equiv t' \quad U,V \vdash t' \equiv t''}{U,V \vdash t \equiv t''}$$

  - Axiom

$$\frac{}{U,V \vdash t \equiv t'} \ (U,V \vdash t \equiv t') \in E$$

  - Substitution

$$\frac{U,W \vdash t \equiv t' \quad W,V \vdash s \equiv s'}{U,V \vdash t[s] \equiv t'[s']}$$

  - Tensor extension

$$\frac{U,V \vdash t \equiv t'}{W \otimes U, W \otimes V \vdash \langle W \rangle t \equiv \langle W \rangle t'}$$

  - Local character

$$\frac{U_i, V \vdash t \circ e_i \equiv t' \circ e_i \ (i \in I)}{U,V \vdash t \equiv t'} \quad \{e_i : U_i \to U\}_{i \in I} \text{ jointly epi}$$

- Soundness

$U,V \vdash t \equiv t'$ is provable from E using TEL

$\Longrightarrow$ $U,V \vdash t \equiv t'$ is satisfied by all models of $\mathcal{C} \circlearrowleft \Sigma_{\text{with } E}$

Internal Completeness

For $\mathbb{S} = \mathscr{C}^{\partial\Sigma}$ with E

$U, V \vdash t \equiv t'$ is satisfied by all models of $\mathbb{S}$

$\Leftrightarrow$ $U, V \vdash t \equiv t'$ is satisfied by the free $\mathbb{S}$-algebra on V

$\left(T_{\mathbb{S}}V,\ \tau^{\mathbb{S}}_V : \Sigma(T_{\mathbb{S}}V) \to (T_{\mathbb{S}}V) \right)$

$\Leftrightarrow$ $U \underset{t'}{\overset{t}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\mathbb{S}} V$

under the assumption that $\mathbb{S}\text{-Alg} \underset{I}{\hookrightarrow} \Sigma\text{-Alg}$ exists

Towards complete rewriting system for TES

The set of axioms E

$$\begin{tikzcd}
 & \Sigma T_\Sigma V \arrow[r, two heads] \arrow[d, "\tau^\Sigma_V"'] \arrow[dr] & \Sigma(T_\Sigma V)_1 \arrow[r, two heads] \arrow[dr] \arrow[d, phantom, "\mathrm{p.o.}"] & \Sigma(T_\Sigma V)_2 & \cdots & \Sigma(T_S V) \arrow[d, "\tau^S_V"] \\
{[C,TV]\otimes A} \arrow[r, shift left] \arrow[r, shift right] & T_\Sigma V \arrow[r, two heads, "\mathrm{coeq}"] \arrow[rrrr, bend right, "q_V"'] & (T_\Sigma V)_1 \arrow[r, two heads] & (T_\Sigma V)_2 & \cdots & T_S V
\end{tikzcd}$$

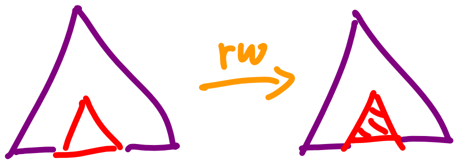
e.g. Theory of groups G $\quad (T_G V = T_\Sigma V/_\equiv)$

$$\frac{}{m(t,e)\equiv t}\; t\in T_\Sigma V \qquad \frac{}{m(i(t),t)\equiv e}\; t\in T_\Sigma V \qquad \frac{}{m(m(t_1,t_2),t_3)\equiv m(t_1,m(t_2,t_3))}\; t_1,t_2,t_3\in T_\Sigma V$$

$$\frac{}{e\equiv e} \qquad \frac{t\equiv s}{i(t)\equiv i(s)} \qquad \frac{t_1\equiv s_1 \quad t_2\equiv s_2}{m(t_1,t_2)\equiv m(s_1,s_2)}$$

$$\boxed{1 \underset{r}{\overset{l}{\rightrightarrows}} T_\Sigma V \longrightarrow T_\Sigma V/_\equiv \;\Leftrightarrow\; [l]_\equiv = [r]_\equiv \text{ in } T_\Sigma V/_\equiv \;\Leftrightarrow\; l\equiv r \text{ is derivable}}$$

$\equiv$ in $T_\Sigma V \rightsquigarrow$ Bidirectional rewriting in $T_\Sigma V$



Concluding Remarks

- Applications
  - Algebraic theories $(\mathbf{Set})$
  - Binding Term Equational Logic (Hamana) $(\mathbf{Set}^{\mathbb{I}})$
  - Nominal Equational Logic (Gabbay & Mathijssen, Clouston & Pitts) $(\mathbf{Nom})$
  - Cominatory Reduction System (Klop) $(\mathbf{Set}^{\mathbb{F}})$
    Second-order abstract syntax (Fiore)
- Further work
  - Abstract condition for confluence
  - Abstract condition for termination