Algebraic theories in the presence of binding operators, substitution, etc.

Chung Kil Hur
Joint work with Marcelo Fiore

Computer Laboratory
University of Cambridge

20th March 2006

Overview

- First order syntax: modelled by algebras on Set
- First order syntax with bindings: modelled by subst-algebras (binding-algebras) on $Set^{\mathbb{F}}$
- First order syntax with equations: modelled by algebraic theories (lawvere theories, or variety of algebras).
- First oder syntax with bindings and equations: equational theories by Kelly and Power (problematic).
- New equational theories.

Definition and Example

Signature for First Order Syntax

$\mathcal{S} = (O, a)$

O is a set of operators

a is an arity function from O to $\mathbb{N}$

Terms built from Signature

$$
\begin{array}{llll}
Term_V & := & V & V \text{ is a set of variables} \\
 & \mid & f(Term_V^1, \ldots, Term_V^n) & f \in O, a(f) = n
\end{array}
$$

Example: Group

$O = \{e, i, m\}$

$a(e) = 0, a(i) = 1, a(m) = 2$

$V = \{x, y, z, \ldots\}$

$Term_V = \{x, e, m(x, m(y, z)), m(e, m(i(x), y)), \ldots\}$

Categorical Construction

- Base Category: Set of sets and functions.
- Endofunctor: $\Sigma X = \coprod_{f \in O} X^{a(f)}$
- Example: Group $\Sigma X = 1 + X + X^2$

State	Set	Depth	Group
S_0	$\emptyset$		$\emptyset$
S_1	$V + \Sigma S_0$	0	$x, y, z, \ldots, e$
S_2	$V + \Sigma S_1$	≤ 1	$x, e, i(e), i(x), m(e, x), \ldots$
S_3	$V + \Sigma S_2$	≤ 2	$x, e, i(x), m(e, x), m(x, i(e)), \ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_∞	Term_V	$< \infty$	$\ldots, m(i(i(x)), m(x, i(e))), \ldots$

- Free Monad of Σ: $TX = \mu A.X + \Sigma A$ with $X + \Sigma TX \xrightarrow{\cong [\eta_X, \tau_X]} TX$

Algebras

Definition of Σ-alg

- objects: $(X, s : \Sigma X \to X)$ where X, s in Set
- morphisms: $f : (X, s) \to (Y, t)$ where $f : X \to Y$ in Set satisfying

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[f]{} & Y \end{array}$$

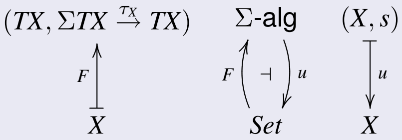
Example: Group

- objects: $(X, 1 + X + X^2 \xrightarrow{[e,i,m]} X)$
- morphisms: $f : (X, e, i, m) \to (Y, e', i', m')$ where $f : X \to Y$

$$\begin{array}{ccc} 1 & \xrightarrow{id} & 1 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} X^2 & \xrightarrow{f^2} & Y^2 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ X & \xrightarrow[f]{} & Y \end{array}$$

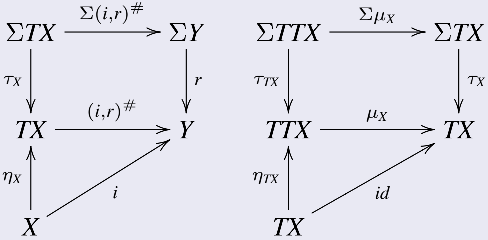
Free Algebras

Free Algebras and Free monads



Free monad $T = uF$

Universal property



1. **Abstract Syntax with Binding**
   - First order syntax without binding
   - First order syntax with binding
   - Substitution Algebras
2. **Equational Theories**
   - Classical approach
   - New view on equations
   - Examples of equations

Definition and Example

Signature for First Order Syntax with Binding

$\mathcal{S} = (O, a)$

O is a set of operators

a is an arity function from O to $\mathbb{N}^*$, *i.e.*$a(f) = (a_1, \ldots, a_k)$

Terms built from Signature

$$
\begin{array}{rcl}
Term(\{x_1, \ldots, x_n\}) & := & \{x_1, \ldots, x_n\} \\
& | & f(x_{n+1}, \ldots, x_{n+a_1}.Term(\{x_1, \ldots, x_{n+a_1}\}), \\
& & \vdots \\
& & x_{n+1}, \ldots, x_{n+a_k}.Term(\{x_1, \ldots, x_{n+a_k}\})) \\
& & \text{where } f \in O, a(f) = (a_1, \ldots, a_k)
\end{array}
$$

Example: Untyped λ-Calculus

$O = \{lam, app\}$

$a(lam) = (1), a(app) = (0, 0)$

$Term(x_1, x_2) = \{x_1, app(x_1, x_2), lam(x_3.app(x_1, app(x_2, x_3))) \ldots\}$

$Term(\{x_1, \ldots, x_n\}) \cong \{[t]_\alpha \mid FV(t) \subseteq \{x_1, \ldots, x_n\}$ in untyped λ-calculus$\}$

Category of Presheaves

Definitions

$\mathbb{F}$	objects	finite cardinals, *i.e.*$\emptyset, \{x_1\}, \{x_1, x_2\}, \ldots$
		let n be $\{x_1, \ldots, x_n\}$
	morphisms	set functions from n to m
$Set^{\mathbb{F}}$	objects	functors from $\mathbb{F}$ to Set
	morphisms	natural transformations between functors

Interpretations

$X \in Set^{\mathbb{F}}$: a language.

$X(n)$: set of terms in context $\{x_1, \ldots, x_n\}$, *i.e.*$\{t \mid x_1, \ldots, x_n \vdash t\}$

$X(n) \xrightarrow{X(\rho)} X(m)$: $x_1, \ldots, x_n \vdash t \mapsto x_1, \ldots, x_m \vdash t[x_{\rho(1)}/x_1, \ldots, x_{\rho(n)}/x_n]$

Presheaf of variables and Type constructor for context extension

$V \in Set^{\mathbb{F}}$: $V(n) = \{x_1, \ldots, x_n\}$, $V(\rho) = \rho$

$\delta : Set^{\mathbb{F}} \to Set^{\mathbb{F}}$: $(\delta X)(n) = X(n+1)$, $(\delta X)(\rho) = X(\rho + 1)$

Categorical Construction

- Base Category: $Set^{\mathbb{F}}$
- Endofunctor: $\Sigma(X) \stackrel{\text{def}}{=} \coprod_{\substack{f \in O \\ a(f)=(a_i)_{1\leq i\leq k}}} \prod_{1\leq i\leq k} \delta^{a_i} X$
- Example: λ-calculus $\Sigma X = \delta X + X^2$

State	Set	$\emptyset$	$\{x_1\}$	$\{x_1, x_2\}$
S_0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
S_1	$V + \Sigma S_0$	$\emptyset$	x_1	x_1, x_2
S_2	$V + \Sigma S_1$	$\lambda x_1.x_1$	$x_1, x_1x_1,$ $\lambda x_2.x_1, \lambda x_2.x_2$	$x_1, x_2, x_1x_2,$ $\lambda x_3.x_3, \ldots$
S_3	$V + \Sigma S_2$	$(\lambda x_1.x_1)(\lambda x_1.x_1),$ $\lambda x_1.\lambda x_2.x_1, \ldots$	$x_1,(x_1x_1)(\lambda x_2.x_2),$ $\lambda x_2.\lambda x_3.x_3, \ldots$	$x_2,(x_1x_2)(\lambda x_3.x_3),$ $\lambda x_3.\lambda x_4.x_3, \ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_∞	$Term$	$Term(\emptyset)$	$Term(\{x_1\})$	$Term(\{x_1, x_2\})$

- Free Monad of Σ: $TX = \mu A.X + \Sigma A$ with $X + \Sigma TX \xrightarrow{\cong [\eta_X, \tau_X]} TX$

Algebras

Example: λ-calculus

- objects: $(X, \delta X + X^2 \xrightarrow{[lam, app]} X)$
- morphisms: $f : (X, lam, app) \to (Y, lam', app')$ where $f : X \to Y$

$$
\begin{array}{ccc}
\delta X & \xrightarrow{\delta f} & \delta Y \\
{\scriptstyle lam}\downarrow & & \downarrow{\scriptstyle lam'} \\
X & \xrightarrow[f]{} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
X^2 & \xrightarrow{f^2} & Y^2 \\
{\scriptstyle app}\downarrow & & \downarrow{\scriptstyle app'} \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

Universality

$$
\begin{array}{ccc}
\delta TV + TV^2 & \xrightarrow{\delta(i, lam, app)^{\#} + (i, lam, app)^{\#2}} & \delta Y + Y^2 \\
{\scriptstyle \tau_V}\downarrow & & \downarrow{\scriptstyle [lam, app]} \\
TV & \xrightarrow{(i, lam, app)^{\#}} & Y \\
{\scriptstyle \eta_V}\uparrow & \nearrow{\scriptstyle i} & \\
V & &
\end{array}
$$

1. **Abstract Syntax with Binding**
   - First order syntax without binding
   - First order syntax with binding
   - Substitution Algebras

2. **Equational Theories**
   - Classical approach
   - New view on equations
   - Examples of equations

Motivations for substitution as an algebraic operation

- We can express operations involving substitution, *e.g.*β equations in λ-calculus.
- It ensures semantics substitution lemma, *i.e.*compositionality of substitution.
- It automatically verifies some structural rules related to substitution, *e.g.*syntactic substitution lemma and admissibility of cut.

Substitutions on concrete terms

$$\sigma_{k,n} : (x_1, \ldots, x_k \vdash t); (x_1, \ldots, x_n \vdash u_1, \ldots, u_k)$$
$$\longmapsto x_1, \ldots, x_n \vdash t[u_1/x_1, \ldots, u_k/x_k]$$

Example: λ-calculus

$$\sigma_{k,n}(t[x_1, \ldots, x_k]; u_1, \ldots, u_k) =$$

$$\begin{array}{lll} & u_i & \text{if } t = x_i \\ | & (\sigma_{k,n}(t_1; u_1, \ldots, u_k))(\sigma_{k,n}(t_2; u_1, \ldots, u_k)) & \text{if } t = t_1 t_2 \\ | & \lambda x_{n+1}.\sigma_{k+1,n+1}(t_1[x_1, \ldots, x_{k+1}]; u_1, \ldots, u_k, x_{n+1}) & \text{if } t = \lambda x_{k+1}.t_1[x_1, \ldots, x_{k+1}] \end{array}$$

Categorical Construction

- Substitution: Function from Substitutable Pairs to Term
- Substitutable Pair: $(x_1, \ldots, x_k \vdash t)$; $(x_1, \ldots, x_n \vdash u_1, \ldots, u_k)$

Type Constructor for Substitutable Pairs

$$(X \bullet Y)(\{x_1, \ldots, x_n\}) \stackrel{\text{def}}{=} \coprod_{k \in \mathbb{N}} X(\{x_1, \ldots, x_k\}) \times (Y(\{x_1, \ldots, x_n\}))^k / \approx$$

$$= \{(t; \vec{u}) | k \in \mathbb{N}, t \in X(\{x_1, \ldots, x_k\}), \vec{u} \in (Y(\{x_1, \ldots, x_n\}))^k\} / \approx$$

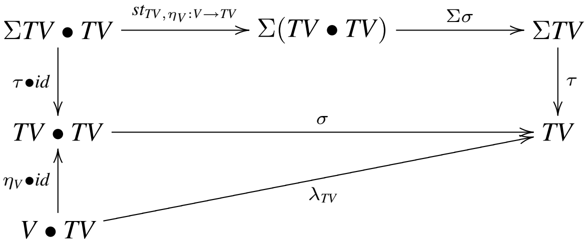
where $\approx$ is the equivalence relation generated by $\sim$

$$(t; u_1, \ldots, u_k) \sim (t'; u'_1, \ldots, u'_{k'}) \quad \text{iff} \quad \exists_{r:k \to k'} X(r)(t) = t' \wedge u_i = u'_{r(i)}$$

The tensor $\bullet : Set^{\mathbb{F}} \times Set^{\mathbb{F}} \to Set^{\mathbb{F}}$ forms a closed monoidal structure with a unit V and suitable natural transformations α, λ, ρ.

Categorical Construction

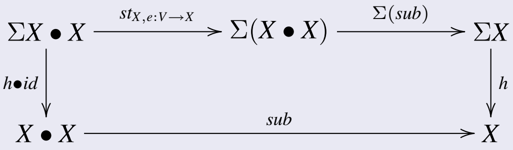
- $st_{X,e:V\to Y} : \Sigma X \bullet Y \to \Sigma(X \bullet Y)$ is a natural transformation between two functors of type $Set^{\mathbb{F}} \times V{\downarrow}Set^{\mathbb{F}} \to Set^{\mathbb{F}}$.
- Example: λ-calculus: $st_{X,e:V\to Y} : (\delta X + X \times X) \bullet Y \cong (\delta X) \bullet Y + (X \times X) \bullet Y \to \delta(X \bullet Y) + (X \bullet Y) \times (X \bullet Y)$
- $\sigma : X \bullet X \to X$ is defined in the following way.



Initial Algebra Semantics

Category of $\mathcal{S}$-subst-algebras

- Objects: $(X, e : V \to X, h : \Sigma X \to X, sub : X \bullet X \to X)$ where (X, e, sub) is a monoid in $Set^{\mathbb{F}}$ and (X, h) is an Σ-algebra such that the following diagram commutes.



- Morphisms: maps of $Set^{\mathbb{F}}$ which are both Σ-algebra and monoid homomorphisms.
- Initial Object: $(TV, \eta_V : V \to TV, \tau : \Sigma TV \to TV, \sigma : TV \bullet TV \to TV)$

Abstract Theory

Definition (Category of Σ-subst-algebras)

- Monoidal closed category $\mathcal{C} = (\mathcal{C}, \otimes, I)$
- Strong endofunctor Σ with a strength[a] st
- Objects: $X = (X, e, h, sub)$, where (X, e, sub) is a monoid in $\mathcal{C}$ and (X, h) is a Σ-algebra such that the following diagram commutes.

$$\begin{array}{ccccc}
\Sigma X \bullet X & \xrightarrow{st_{X,X}} & \Sigma(X \bullet X) & \xrightarrow{\Sigma(sub)} & \Sigma X \\
{\scriptstyle h \bullet id}\downarrow & & & & \downarrow{\scriptstyle h} \\
X \bullet X & & \xrightarrow{\qquad\qquad sub \qquad\qquad} & & X
\end{array}$$

- Morphisms: maps of $\mathcal{C}$ which are both Σ-algebra and monoid homomorphisms.

[a] A strength for Σ is a natural transformation of type $\Sigma(X) \otimes Y \to \Sigma(X \otimes Y)$ satisfying some coherence conditions.

Abstract Theory

Theorem

If Σ has a free monad T then $(TI, \eta_I, \sigma, \tau)$ is an initial Σ-subst-algebra, where $\tau : \Sigma TI \to TI$ is the free Σ-algebra of I and σ is given as follows:

$$\sigma : \; TI \bullet TI \xrightarrow{\bar{st}_{I,TI}} T(I \bullet TI) \xrightarrow{T\lambda_{TI}} TTI \xrightarrow{\mu_I} TI \; ,$$

where $\bar{st} : TX \bullet Y \to T(X \bullet Y)$ is the lifting of the strength st of Σ to its free monad T.

Abstract Theory

Definition (Category of Σ-ptd-subst-algebras)

- Monoidal closed category $\mathcal{C} = (\mathcal{C}, \otimes, I)$
- u-strong endofunctor Σ with a u-strength[a] st for the forgetful functor u from $I{\downarrow}\mathcal{C}$
- Objects: $X = (X, e, h, sub)$, where (X, e, sub) is a monoid in $\mathcal{C}$ and (X, h) is a Σ-algebra such that the following diagram commutes.

$$\begin{array}{ccccc} \Sigma X \bullet X & \xrightarrow{st_{X,e:I\to X}} & \Sigma(X \bullet X) & \xrightarrow{\Sigma(sub)} & \Sigma X \\ {\scriptstyle h\bullet id}\downarrow & & & & \downarrow{\scriptstyle h} \\ X \bullet X & & \xrightarrow{\qquad sub \qquad} & & X \end{array}$$

- Morphisms: maps of $\mathcal{C}$ which are both Σ-algebra and monoid homomorphisms.

[a] A u-strength for Σ is a natural transformation of type $\Sigma(X) \otimes uY \to \Sigma(X \otimes uY)$ satisfying some coherence conditions.

Abstract Theory

Theorem

$(TI, \eta_I, \sigma, \mu_I)$ is an initial Σ-ptd-subst-algebra, where σ is given by the following composite:

$$TI \otimes TI = TI \otimes u(\eta_I : I \to TI) \xrightarrow{st_{I,\eta_I}} T(I \otimes u(\eta_I)) = T(I \otimes TI)$$
$$\xrightarrow{T\lambda_{TI}} TTI \xrightarrow{\mu_I} TI \ .$$

1. **Abstract Syntax with Binding**
   - First order syntax without binding
   - First order syntax with binding
   - Substitution Algebras

2. **Equational Theories**
   - Classical approach
   - New view on equations
   - Examples of equations

Motivation for equations

Group Axioms

Given a group algebra (X, e, i, m),

- (G1) $\forall x, y, z \in X \quad m(m(x, y), z) = m(x, m(y, z))$
- (G2) $\forall x \in X \quad m(e, x) = x$
- (G3) $\forall x \in X \quad m(i(x), x) = e$

β, η equations for λ-calculus

Given a λ-calculus algebra (X, e, lam, app, sub),

- (β-eqn)
  $\forall t \in \delta X(n), u \in X(n) \quad app(lam(x_{n+1}.t), u) = sub(t; (x_1, \ldots, x_n, u))$
- (η-eqn)
  $\forall t \in X(n) \quad lam(x_{n+1}.app(t, x_{n+1})) = t$

Kelly & Power equational theories

KP Theory	Groups
Sig $S : \lvert\mathcal{C}_{\mathsf{fp}}\rvert \to \mathcal{C}$ $\Sigma X = \coprod_{a\in\lvert\mathcal{C}_{\mathsf{fp}}\rvert} \mathcal{C}(a,X) \otimes S(a)$ $TX = \mu A.X + \Sigma A$ $TX \cong$ all terms freely generated from X (Equation) $D : \lvert\mathcal{C}_{\mathsf{fp}}\rvert \to \mathcal{C}$ together with $\forall\, a \in \lvert\mathcal{C}_{\mathsf{fp}}\rvert \;\; D(a) \overset{E_L(a)}{\underset{E_R(a)}{\rightrightarrows}} T(a)$	Sig $S : \mathbb{N} \to \mathit{Set}$ $(0,1,2) \mapsto (1,1,1)$ $\Sigma X = \coprod_{a\in\mathbb{N}} X^a \times S(a) = 1 + X + X^2$ $TX = \mu A.X + \Sigma A$ $TX \cong$ all group terms freely generated from a variable set X (Equation) $D : \mathbb{N} \to \mathit{Set} : (1,3) \mapsto (2,1)$ $D(1) = 2 \overset{E_L(1)}{\underset{E_R(1)}{\rightrightarrows}} T(1)$ $E_L(1)(1) = m(e,x_1), E_R(1)(1) = x_1$ $E_L(1)(2) = m(i(x_1),x_1), E_R(1)(2) = e$ $D(3) = 1 \overset{E_L(3)}{\underset{E_R(3)}{\rightrightarrows}} T(3)$ $E_L(3)(1) = m(m(x_1,x_2),x_3),$ $E_R(3)(1) = m(x_1,m(x_2,x_3))$

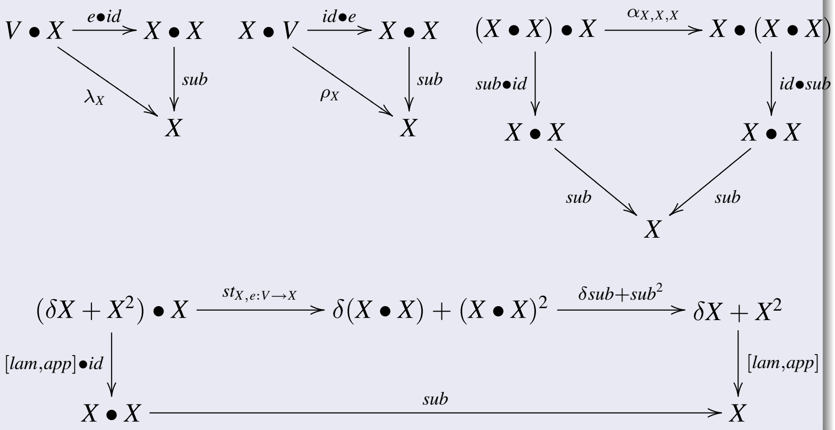
Free Constructions

KP Theory	Groups
There exists a finitary monad T' such that $T'\text{-Alg} \cong \Sigma\text{-alg}/_{E_L=E_R}$ $F \uparrow \dashv \downarrow u$ $\mathcal{C}$ where Σ-alg$/_{E_L=E_R}$ is a full subcategory of Σ-alg whose objects are Σ-algebras satisfying the equations.	There exists a finitary monad T' such that $T'\text{-Alg} \cong \Sigma\text{-alg}/_{E_L=E_R}$ $F \uparrow \dashv \downarrow u$ Set where Σ-alg$/_{E_L=E_R}$ is a full subcategory of Σ-alg whose objects are group-algebras, *i.e.*(X, e, i, m) satisfying the group axioms $G1$, $G2$, and $G3$

Problems with applying K.P. theory to subst-algebras

Recall: subst-algebras for λ-calculus

Objects: $(X, e : V \to X, lam : \delta X \to X, app : X^2 \to X, sub : X \bullet X \to X)$ satisfying



1. Abstract Syntax with Binding
   - First order syntax without binding
   - First order syntax with binding
   - Substitution Algebras

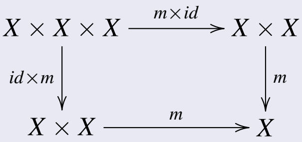
2. Equational Theories
   - Classical approach
   - New view on equations
   - Examples of equations

A new view on equational theories: Observation

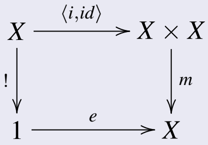
Example: Group

Given a group algebra $(X, 1 + X + X^2 \xrightarrow{[e,i,m]} X)$,

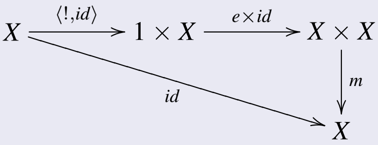
- (G1) $\forall x, y, z \in X \quad m(m(x, y), z) = m(x, m(y, z))$
- (G2) $\forall x \in X \quad m(e, x) = x$
- (G3) $\forall x \in X \quad m(i(x), x) = e$



G1



G3



G2

A new view on equational theories

Observation

$$
\begin{array}{ccc}
\Sigma\text{-alg} & & D\text{-alg} \\
1 + X + X^2 & & X^3 + X + X \\
\Big\downarrow{\scriptstyle [e,i,m]} & \longmapsto & {\scriptstyle E_L}\Big\downarrow\Big\downarrow{\scriptstyle E_R} \\
X & & X
\end{array}
$$

Definition (Equations for algebras)

- a category $\mathcal{C}$ and an endofunctor Σ
- Equation: a domain endofunctor D with functors E_L and E_R from Σ-alg to D-alg preserving base-objects of algebras, *i.e.* $u'E_L = u$ and $u'E_R = u$.

$$
\begin{array}{ccc}
\Sigma\text{-alg} & \overset{E_L}{\underset{E_R}{\rightrightarrows}} & D\text{-alg} \\
{\scriptstyle u}\Big\downarrow & \swarrow{\scriptstyle u'} & \\
\mathcal{C} & &
\end{array}
$$

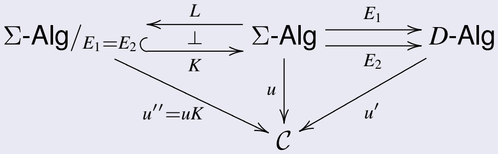
Free constructions

Theorem

If either

- (Cond1) $\mathcal{C}$ is cocomplete and Σ, D preserve colimits of ω-chains.
- (Cond2) $\mathcal{C}$ is cocomplete and well-copowered, and Σ preserves epimorphisms.

then



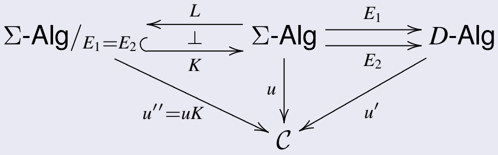
Free constructions

Theorem

If either

- (Cond1) $\mathcal{C}$ has coequalizers, pushouts and colimits of chains whose cardinals are less than or equal to a regular cardinal λ, and Σ, D preserve colimits of chains whose cardinals are equal to λ.
- (Cond2) $\mathcal{C}$ has coequalizers, pushouts and colimits of chains (whose cardinals are less than a regular cardinal λ) and is (λ-)well-copowered, and Σ preserves epimorphisms.

then



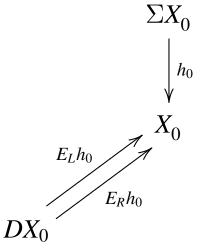
Sketch of the Proof

$$\begin{array}{c} \Sigma X \\ \downarrow{\scriptstyle h} \\ X \end{array}$$

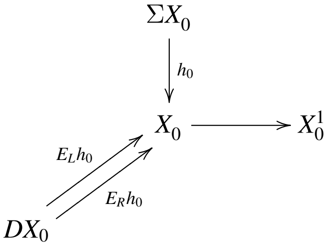
Sketch of the Proof

$$\begin{array}{c} \Sigma X_0 \\ \Big\downarrow {\scriptstyle h_0} \\ X_0 \end{array}$$

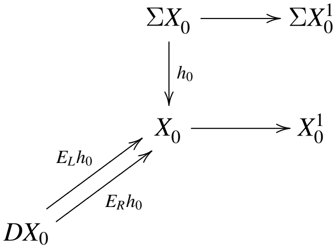
Sketch of the Proof



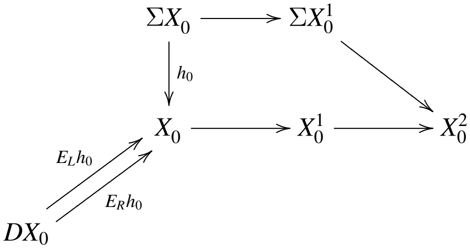
Sketch of the Proof



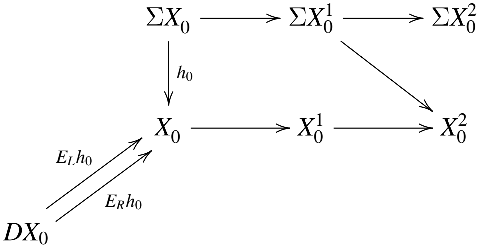
Sketch of the Proof



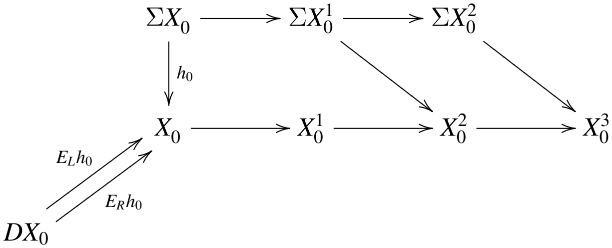
Sketch of the Proof



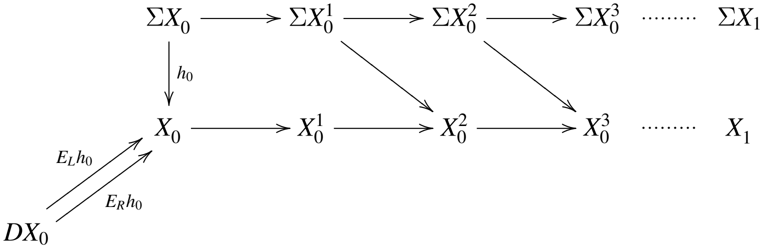
Sketch of the Proof



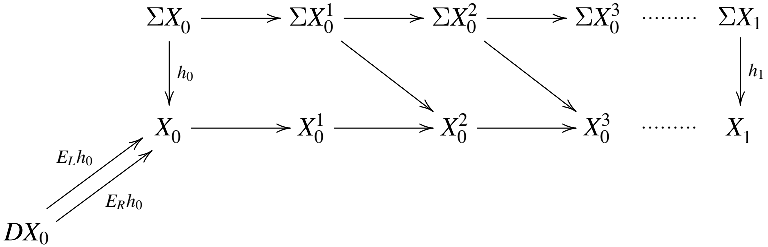
Sketch of the Proof



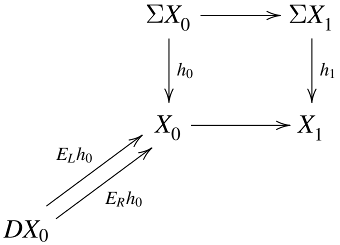
Sketch of the Proof



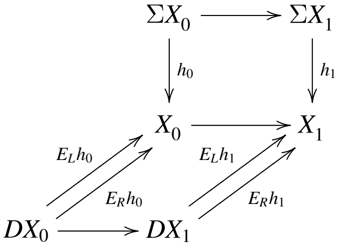
Sketch of the Proof



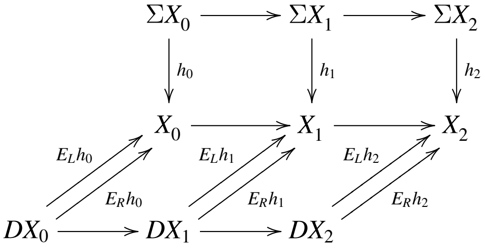
Sketch of the Proof



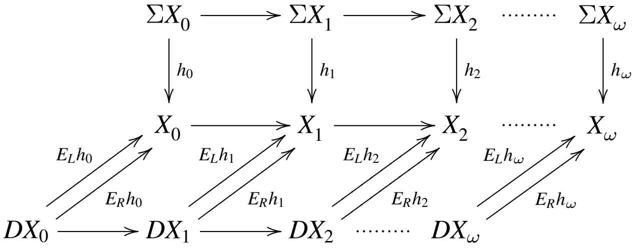
Sketch of the Proof



Sketch of the Proof



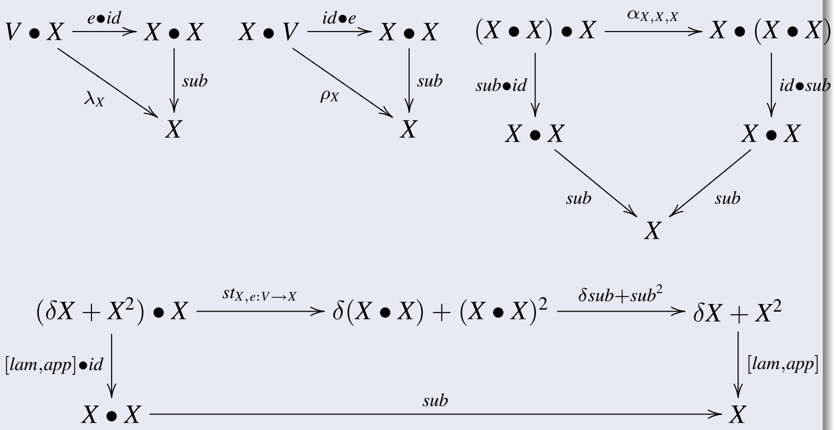
Sketch of the Proof



Examples of Equations

subst-algebras for λ-calculus

Objects: $(X, e : V \to X, lam : \delta X \to X, app : X^2 \to X, sub : X \bullet X \to X)$ satisfying

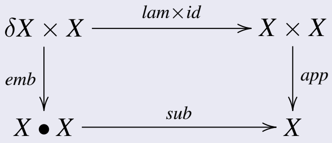


Examples of Equations

Example: λ-calculus

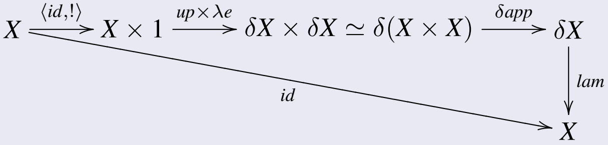
(β-eqn)

$\forall\, t \in \delta X(n), u \in X(n) \quad app(lam(x_{n+1}.t), u) = sub(t; (x_1, \ldots, x_n, u))$



(η-eqn)

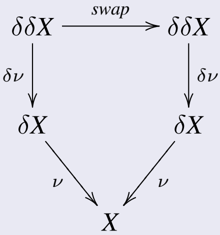
$\forall\, t \in X(n) \quad lam(x_{n+1}.app(t, x_{n+1})) = t$



Examples of Equations

π-calculus

- Base category $Set^{\mathbb{I}}$
- $\nu : \delta X \to X$, where $\delta X \cong (V \multimap X)$
- $\nu x.\nu y.t = \nu y.\nu x.t$
- $\forall\, t \in \delta^2 X(n) \quad \nu x_{n+1}.\nu x_{n+2}.t = \nu x_{n+1}.\nu x_{n+2}.t[x_{n+2}/x_{n+1}, x_{n+1}/x_{n+2}]$



Future Work

- Modelling DILL (Dual Intuitionistic Linear Logic)
- Modelling Dependent type languages.
- Develop the equational theories in pre-ordered setting, and apply them to Term Rewriting Systems.
- Find more applications of the equational theories. Example: the free-algebra models for the π-calculus by Ian Stark.