

# Term Equational Rewrite Systems and Logics

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# Outline of the talk

- 1 General abstract **Definition** of Term Equational Rewrite System (TERS).
- 2 Development of the **Theory** of Term Equational Rewrite Systems.
- 3 **Applications** of Term Equational Rewrite Systems.

# Overview: Definition of TERS

Term Equational Rewrite Systems are a **framework** for developing systems/logics of equations and rewrites.

① **Equational logic**

reasoning about equality between terms.

② **Rewriting system/logic**

deriving/reasoning about rewrite relation between terms.



- terms
- equational/rewrite judgements
- algebraic models

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$$\text{FM}(\mathcal{A}) \models \Gamma \vdash t \star t'$$

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$\Uparrow$  Internal Completeness

$$\text{FM}(\mathcal{A}) \models \Gamma \vdash t \star t'$$

e.g., standard term rewriting modulo equations

$\mathcal{M}(\mathcal{A})$

(Hore & Hur, ICALP 07)

$$\mathcal{A} \xrightarrow{\text{compact logic}} \Gamma \vdash t \star t'$$

# Overview: Applications of TERS

- 1 Equational logic
  - algebraic theories and first-order equational logic
  - **Nominal Equational Logic**  
(Gabbay & Mathijssen 06; Clouston & Pitts 07)
- 2 Rewriting system/logic
  - first-order Term Rewriting System
  - Binding Term Rewriting System (Hamana 03)
  - Nominal Rewriting System (Fernández, Gabbay, Mackie 04)
  - Combinatory Reduction System (Klop 80)

In this talk, in order to convey the basic ideas more easily, we consider TERSs **in restricted form** only dealing with

- equations (not rewriting)
- single-sorted (not multi-sorted)

# Definition: I. Terms and Judgements

- Symmetric monoidal closed category  $(\mathcal{D}, I, \otimes, [-, =])$
- Strong monad  $\mathbb{T} = (T, \eta, \mu, \tau)$

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$$TV \in \mathcal{D}$$

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- 1 **Generalised Term**  $t : U \rightarrow TV$ , denoted  $U, V \vdash t$

Parameter (e.g., Atoms, Object Variables)

- 2 **Generalised Equation**  $U, V \vdash t \equiv t'$

## Definition: II. Algebraic Models



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- Satisfaction relation

$$(D, \xi) \models U, V \vdash t \equiv t'$$

$\Updownarrow$  Def

$$[V, D] \otimes U \xrightarrow[\llbracket t' \rrbracket_{(D, \xi)}]{\llbracket t \rrbracket_{(D, \xi)}} D$$

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Evaluation

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where

$$\llbracket t \rrbracket_{(D, \xi)} : [V, D] \otimes U \xrightarrow{id \otimes t} [V, D] \otimes TV \xrightarrow{\tau} T([V, D] \otimes V) \xrightarrow{T\xi} TD \xrightarrow{\xi} D$$

$$\mathcal{A} \xrightarrow{\text{TERL}} U, V \vdash t \equiv t'$$

# Theory: TERL

- Equivalence relation

$$\frac{}{U, V \vdash t \equiv t} \quad \frac{U, V \vdash t \equiv t'}{U, V \vdash t' \equiv t} \quad \frac{U, V \vdash t \equiv t' \quad U, V \vdash t' \equiv t''}{U, V \vdash t \equiv t''}$$

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- Local Character

$$\frac{U_i, V \vdash t \circ e_i \equiv t' \circ e_i \quad (i \in I)}{U, V \vdash t \equiv t'} \quad \{e_i : U_i \rightarrow U\}_{i \in I} \text{ jointly epi}$$

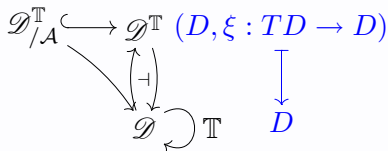
# Theory: Soundness of TERL

$$\mathcal{A} \xrightarrow{\text{TERL}} U, V \vdash t \equiv t'$$



$$\forall_{(D, \xi)} (D, \xi) \models \mathcal{A} \implies (D, \xi) \models U, V \vdash t \equiv t'$$

# Theory: Internal Completeness



$$\mathcal{A} \models U, V \vdash t \equiv t'$$

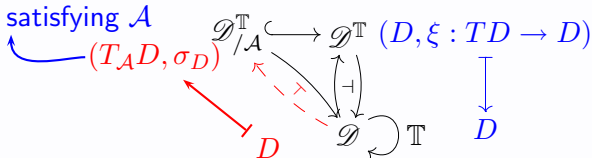
$\Downarrow$  Def

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free model satisfying  $\mathcal{A}$



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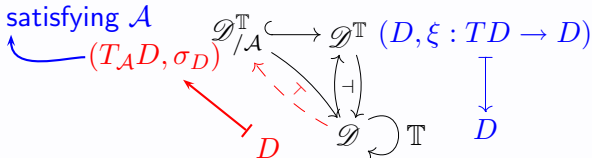
$$\forall_{(D, \xi)} (D, \xi) \models \mathcal{A} \implies (D, \xi) \models U, V \vdash t \equiv t'$$

$\Downarrow$

$$(T_{\mathcal{A}}V, \sigma_V) \models U, V \vdash t \equiv t'$$

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$\Downarrow$

$$q_V \circ t = q_V \circ t' : U \xrightarrow[t']{t} TV \xrightarrow{q_V} T_{\mathcal{A}}V$$

where

$q_V : TV \rightarrow T_{\mathcal{A}}V$  is the quotient of  $TV$  under  $\mathcal{A}$ .

# Theory: Construction of $q_V$

The theory of **Equational Systems** (Fiore & Hur, ICALP 07) provides an **explicit construction** of the quotient map

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If

- $\mathcal{D}$  is cocomplete
- The endofunctors  $T$  and  $[W, -] \otimes R$  for every  $(R, W \vdash s \equiv s') \in \mathcal{A}$  preserve colimits of  $\omega$ -chains and epimorphisms.

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then

$$\begin{array}{ccccccc}
 \forall (R, W \vdash s \equiv s') \in \mathcal{A} & & TTV & \xrightarrow{\gg Tq_1} & TY_1 & \xrightarrow{\gg Tq_2} & TY_2 \longrightarrow \cdots \longrightarrow TTA_V \\
 & \vdots & \downarrow \mu_V & \searrow & \text{po} & \searrow & \downarrow \sigma_V \\
 [W, TV] \otimes R & \xrightarrow{\gg \llbracket s \rrbracket_{(TV, \mu_V)}} & TV & \xrightarrow{\gg q_1} & Y_1 & \xrightarrow{\gg q_2} & Y_2 \longrightarrow \cdots \longrightarrow T_{\mathcal{A}}V \text{ colim} \\
 & \vdots & \downarrow \llbracket s' \rrbracket_{(TV, \mu_V)} & \searrow & & \searrow & \\
 & & & & & & q_V
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$$q_V : TV \rightarrow T_{\mathcal{A}}V .$$

If

- $\mathcal{D}$  is cocomplete
- The endofunctors  $\Sigma$  and  $[W, -] \otimes R$  for every  $(R, W \vdash s \equiv s') \in \mathcal{A}$  preserve colimits of  $\omega$ -chains and epimorphisms.
- $\mathbb{T}$  is a free monad of an endofunctor  $\Sigma$ .

then

$$\begin{array}{ccccccc}
 \forall (R, W \vdash s \equiv s') \in \mathcal{A} & \Sigma TV & \xrightarrow{\Sigma q_1} & \Sigma Y_1 & \xrightarrow{\Sigma q_2} & \Sigma Y_2 & \longrightarrow \cdots \longrightarrow \Sigma T_{\mathcal{A}}V \\
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 \end{array}$$

# Theory: Towards External Completeness

$$\mathcal{A} \models U, V \vdash t \equiv t'$$



$$q_V \circ t = q_V \circ t' : U \begin{array}{c} \xrightarrow{t} \\ \xrightarrow{t'} \end{array} TV \xrightarrow{q_V} T_{\mathcal{A}}V$$



Explicit construction of  $q_V$



We may synthesize a sound and complete logic.

# Application

## Nominal Term Equational System and Logic (NTES & NTEL)



# NTES/L: S.M.C. Category

- 1 *Nom*: the category of nominal sets
- 2  $(1, \#, [-, =])$  : symmetric monoidal closed structure

- 1  $\mathcal{Nom}$ : the category of nominal sets
  - 2  $(1, \#, [-, =])$ : symmetric monoidal closed structure
- $1$ : a singleton nominal set
  - $M \# M' \triangleq \{ (m, m') \in M \times M' \mid \text{supp}(m) \cap \text{supp}(m') = \emptyset \}$
  - $\mathbb{A}$ : the nominal set of atoms
  - $\mathbb{A}^{\#n} = \{ (a_1, \dots, a_n) \in \mathbb{A}^n \mid \forall_{1 \leq i \neq j \leq n} a_i \# a_j \}$
  - $[\mathbb{A}^{\#n}, M] = \{ \langle \vec{a}^n \rangle m \mid (\vec{a}^n) \in \mathbb{A}^{\#n}, m \in M \}$

where

$$\vec{a}^n \triangleq a_1, \dots, a_n$$

$\langle \vec{a}^n \rangle m$  is an  $\alpha$ -equivalence class

e.g.  $\langle a, b \rangle f(a, b, a) = \langle c, d \rangle f(c, d, c)$

# NTES/L: Strong Monad

- Signature  $\Sigma = \{\Sigma(k)\}_{k \in \mathbb{N}}$   
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- Signature  $\Sigma = \{ \Sigma(k) \}_{k \in \mathbb{N}}$   
for  $\Sigma(k)$  a nominal set of operators of arity  $k$
- Endofunctor

$$\begin{aligned} F_{\Sigma}(M) &= \coprod_{k \in \mathbb{N}} \Sigma(k) \times M^k \\ &= \{ \circ m_1 \dots m_k \mid k \in \mathbb{N}, \circ \in \Sigma(k), m_1, \dots, m_k \in M \} \end{aligned}$$

- with Strength

$$\begin{aligned} \tau : \quad M' \# F_{\Sigma}(M) &\rightarrow F_{\Sigma}(M' \# M) \\ (m', \circ m_1 \dots m_k) &\mapsto \circ (m', m_1) \dots (m', m_k) \end{aligned}$$

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- Free monad  $\mathbb{T}_{\Sigma} = (T_{\Sigma}, \eta, \mu)$

$$\begin{aligned} t \in T_{\Sigma}M &::= m && (m \in M) \\ &| \circ t_1 \dots t_k && (\circ \in \Sigma(k), t_1, \dots, t_k \in T_{\Sigma}M) \end{aligned}$$

- with Inductively defined Strength

# NTES/L: Terms and Judgements

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Generalised terms  $U \rightarrow T_{\Sigma} V$

Choose

$$U = \mathbb{A}^{\#n} \quad \text{for } n \in \mathbb{N}$$

$$V = \mathbb{A}^{\#n_1} + \dots + \mathbb{A}^{\#n_k} \quad \text{for } k \in \mathbb{N}, n_1, \dots, n_k \in \mathbb{N}$$

# NTES/L: Terms and Judgements

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Then

$$T_\Sigma V$$
$$x_1 : n_1, \dots, x_k : n_k \vdash t$$

Inductively defined by

$$\frac{}{x_1 : n_1, \dots, x_k : n_k \vdash x_i(a'_1, \dots, a'_{n_i})} \left[ \begin{array}{l} i \in \{1, \dots, k\}, \\ (a'^{n_i}) \in \mathbb{A}^{\#n_i} \end{array} \right]$$
$$\frac{x_1 : n_1, \dots, x_k : n_k \vdash t_i \quad (1 \leq i \leq k)}{x_1 : n_1, \dots, x_k : n_k \vdash \circ t_1 \dots t_k} \left[ \circ \in \Sigma(k) \right]$$



# NTES/L: Terms and Judgements

Generalised terms  $U \rightarrow T_\Sigma V$

Choose

$$U = \mathbb{A}^{\#n} \quad \text{for } n \in \mathbb{N}$$

$$V = \mathbb{A}^{\#n_1} + \dots + \mathbb{A}^{\#n_k} \quad \text{for } k \in \mathbb{N}, n_1, \dots, n_k \in \mathbb{N}$$

Then

$$U \rightarrow T_\Sigma V$$
$$\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash t$$

Inductively defined by

$$\frac{\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash x_i(a'_1, \dots, a'_{n_i})}{\left[ \begin{array}{l} i \in \{1, \dots, k\}, \\ (\vec{a}'^{n_i}) \in \mathbb{A}^{\#n_i}, \{\vec{a}'^{n_i}\} \subseteq \{\vec{a}^n\} \end{array} \right]}$$

$$\frac{\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash t_i \quad (1 \leq i \leq k)}{\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash \circ t_1 \dots t_k} \left[ \begin{array}{l} \circ \in \Sigma(k), \\ \text{supp}(\circ) \subseteq \{a_1, \dots, a_n\} \end{array} \right]$$

# NTES/L: Terms and Judgements

Generalised terms  $U \rightarrow T_{\Sigma}V$

Choose

$$U = \mathbb{A}^{\#n} \quad \text{for } n \in \mathbb{N}$$

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Then

Generalised terms & judgements

$$\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash t$$

$$\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash t \equiv t'$$

Inductively defined by

$$\frac{}{\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash x_i(a'_1, \dots, a'_{n_i})} \left[ \begin{array}{l} i \in \{1, \dots, k\}, \\ (\vec{a}'^{n_i}) \in \mathbb{A}^{\#n_i}, \{\vec{a}'^{n_i}\} \subseteq \{\vec{a}^n\} \end{array} \right]$$

$$\frac{\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash t_i \quad (1 \leq i \leq k)}{\langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash \circ t_1 \dots t_k} \left[ \begin{array}{l} \circ \in \Sigma(k), \\ \text{supp}(\circ) \subseteq \{a_1, \dots, a_n\} \end{array} \right]$$

- Algebras

Eilenberg-Moore algebras for  $\mathbb{T}_\Sigma$

||

$F_\Sigma$ -algebras  $(M, e)$

where

$M \in \mathcal{Nom}$

$e = \{ e_n : \Sigma(n) \times M \rightarrow M \}_{n \in \mathbb{N}}$

# NTES/L: Algebraic Models

- Algebras

Eilenberg-Moore algebras for  $\mathbb{T}_\Sigma$

$\equiv$

$F_\Sigma$ -algebras  $(M, e)$

where

$M \in \mathcal{Nom}$

$e = \{ e_n : \Sigma(n) \times M \rightarrow M \}_{n \in \mathbb{N}}$

- Satisfaction relation

$(M, e) \models \langle \vec{a}^n \rangle x_1 : n_1, \dots, x_k : n_k \vdash t \equiv t'$

$\iff$  Def

$[\mathbb{A}^{\#n_1}, M] \times \dots \times [\mathbb{A}^{\#n_k}, M] \times \mathbb{A}^{\#n} \xrightarrow{\begin{matrix} \llbracket t \rrbracket_{M,e} \\ \parallel \\ \llbracket t' \rrbracket_{M,e} \end{matrix}} M$

# NTES/L: Untyped $\lambda$ -calculus (Example)

- Signature

$$\begin{array}{ll} \Sigma_\lambda(0) = \{V_a \mid a \in \mathbb{A}\} & \pi \cdot V_a = V_{\pi(a)} \\ \Sigma_\lambda(1) = \{L_a \mid a \in \mathbb{A}\} & \pi \cdot L_a = L_{\pi(a)} \\ \Sigma_\lambda(2) = \{A\} & \pi \cdot A = A \end{array}$$

- Axioms

$$\begin{array}{ll} \langle a, b \rangle x : 1 \vdash L_a x(a) \equiv L_b x(b) & (\alpha) \\ \langle a \rangle x : 0 \vdash L_a (A x()) V_a \equiv x() & (\eta) \\ \vdots & \end{array}$$

- Algebras


$$\begin{array}{l} \llbracket V \rrbracket : \mathbb{A} \rightarrow M \\ \llbracket L \rrbracket : \mathbb{A} \times M \rightarrow M \\ \llbracket A \rrbracket : M^2 \rightarrow M \end{array}$$

# NTES/L: Untyped $\lambda$ -calculus (Example)


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- Axioms

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- Algebras

$$\begin{array}{l} \llbracket V \rrbracket : \mathbb{A} \rightarrow M \\ \llbracket L \rrbracket : \mathbb{A} \times M \rightarrow M \\ \llbracket A \rrbracket : M^2 \rightarrow M \end{array}$$


$\llbracket L \rrbracket : [\mathbb{A}, M] \rightarrow M$

# NTES/L: Sound Logic NTEL

Ref

Sym

Trans

Axiom

Ref

Sym

Trans

Axiom

$$\text{Subst} \frac{\langle \vec{a}^n \rangle \Delta \vdash t \equiv t' \quad \{ \langle \vec{b}_x^{\Delta(x)} \rangle \Gamma \vdash s_x \equiv s'_x \}_{x \in |\Delta|}}{\langle \vec{a}^n \rangle \Gamma \vdash t[x \mapsto \langle \vec{b}_x \rangle s_x]_{x \in |\Delta|} \equiv t'[x \mapsto \langle \vec{b}_x \rangle s'_x]_{x \in |\Delta|}}$$



Ref

Sym

Trans

Axiom

$$\text{Subst} \frac{\langle \vec{a}^n \rangle \Delta \vdash t \equiv t' \quad \{ \langle \vec{b}_x^{\Delta(x)} \rangle \Gamma \vdash s_x \equiv s'_x \}_{x \in |\Delta|}}{\langle \vec{a}^n \rangle \Gamma \vdash t[x \mapsto \langle \vec{b}_x \rangle s_x]_{x \in |\Delta|} \equiv t'[x \mapsto \langle \vec{b}_x \rangle s'_x]_{x \in |\Delta|}}$$

## Tensor Extension

$$\text{Intro} \frac{\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t' \quad \vec{b}^m \# \vec{a}^n \quad \{ \langle \vec{c}_x^{\Gamma(x)} \rangle \}_{x \in |\Gamma|} \# \vec{b}^m}{\langle \vec{a}^n, \vec{b}^m \rangle \Gamma^{[m]} \vdash t[x \mapsto \langle \vec{c}_x \rangle x(\vec{c}_x, \vec{b})]_x \equiv t'[x \mapsto \langle \vec{c}_x \rangle x(\vec{c}_x, \vec{b})]_x}$$

where  $|\Gamma^{[m]}| = |\Gamma|$  and  $\forall_{x \in |\Gamma|} \Gamma^{[m]}(x) = \Gamma(x) + m$

Ref

Sym

Trans

Axiom

$$\text{Subst} \frac{\langle \vec{a}^n \rangle \Delta \vdash t \equiv t' \quad \{ \langle \vec{b}_x^{\Delta(x)} \rangle \Gamma \vdash s_x \equiv s'_x \}_{x \in |\Delta|}}{\langle \vec{a}^n \rangle \Gamma \vdash t[x \mapsto \langle \vec{b}_x \rangle s_x]_{x \in |\Delta|} \equiv t'[x \mapsto \langle \vec{b}_x \rangle s'_x]_{x \in |\Delta|}}$$

Tensor Extension

$$\text{Intro} \frac{\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t' \quad \vec{b}^m \# \vec{a}^n \quad \{ \langle \vec{c}_x^{\Gamma(x)} \rangle \}_{x \in |\Gamma|} \# \vec{b}^m}{\langle \vec{a}^n, \vec{b}^m \rangle \Gamma^{[m]} \vdash t[x \mapsto \langle \vec{c}_x \rangle x(\vec{c}_x, \vec{b})]_x \equiv t'[x \mapsto \langle \vec{c}_x \rangle x(\vec{c}_x, \vec{b})]_x}$$

where  $|\Gamma^{[m]}| = |\Gamma|$  and  $\forall_{x \in |\Gamma|} \Gamma^{[m]}(x) = \Gamma(x) + m$

Local Character

$$\text{Elim} \frac{\langle \vec{a}^n, \vec{b}^m \rangle \Gamma \vdash t \equiv t'}{\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t'} \vec{b} \# \vec{a}, t, t'$$

# NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle x_1 : n_1, \dots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \xrightleftharpoons[u]{u} T_{\Sigma} V \xrightarrow{q_V} T_{\Sigma, \mathcal{A}} V$$

where  $V = \mathbb{A}^{\#n_1} + \dots + \mathbb{A}^{\#n_l}$

# NTES/L: Completeness

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$$T_{\Sigma} V = \{ x_1 : n_1, \dots, x_l : n_l \vdash t \}$$

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$$T_{\Sigma} V = \{ x_1 : n_1, \dots, x_l : n_l \vdash t \}$$

$$\forall (\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A}$$

$$\begin{array}{ccccccc}
 & & F_{\Sigma} T_{\Sigma} V & \xrightarrow{F_{\Sigma} q_1} & F_{\Sigma} Y_1 & \xrightarrow{F_{\Sigma} q_2} & F_{\Sigma} Y_2 \cdots F_{\Sigma} T_{\Sigma, \mathcal{A}} V \\
 & \vdots & & & & & \\
 [\tilde{\Gamma}, T_{\Sigma} V] \# \mathbb{A}^{\#n} & \xrightarrow[\llbracket t' \rrbracket]{\llbracket t \rrbracket} & T_{\Sigma} V & \xrightarrow{q_1} & Y_1 & \xrightarrow{q_2} & Y_2 \cdots T_{\Sigma, \mathcal{A}} V^{\text{colim}} \\
 & \vdots & & & & & \\
 & & & \searrow & \text{po} & \searrow & \\
 & & & & & & \downarrow \widehat{\sigma}_V
 \end{array}$$

# NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle x_1 : n_1, \dots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \xrightarrow[u]{u} T_\Sigma V \xrightarrow{q_V} T_{\Sigma, \mathcal{A}} V$$

where  $V = \mathbb{A}^{\#n_1} + \dots + \mathbb{A}^{\#n_l}$

$$T_\Sigma V = \{ x_1 : n_1, \dots, x_l : n_l \vdash t \}$$

$$\begin{array}{ccccccc} \forall (\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} & F_\Sigma T_\Sigma V & \xrightarrow{F_\Sigma q_1} & F_\Sigma Y_1 & \xrightarrow{F_\Sigma q_2} & F_\Sigma Y_2 & \cdots & F_\Sigma T_{\Sigma, \mathcal{A}} V \\ \vdots & \widehat{\mu}_V \downarrow & & \text{po} & & & & \downarrow \widehat{\sigma}_V \\ [\tilde{\Gamma}, T_\Sigma V] \# \mathbb{A}^{\#n} & \xrightarrow[\llbracket t' \rrbracket]{\llbracket t \rrbracket}} & T_\Sigma V & \xrightarrow{q_1} & Y_1 & \xrightarrow{q_2} & Y_2 & \cdots & T_{\Sigma, \mathcal{A}} V \text{ colim} \\ \vdots & & & & \xrightarrow{q_V} & & & & \end{array}$$

$$\text{Ref}^1 \frac{}{t \equiv^1 t} \quad \text{Sym}^1 \frac{t \equiv^1 t'}{t' \equiv^1 t} \quad \text{Trans}^1 \frac{t \equiv^1 t' \quad t' \equiv^1 t''}{t \equiv^1 t''}$$

$$\text{Axiom}^1 \frac{(\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} \quad \vec{b}^n \# \{ \langle \vec{c}_x \rangle s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_\Sigma V] \}_{x \in |\Gamma|}}{((\vec{a} \vec{b}) \cdot t)[x \mapsto \langle \vec{c}_x \rangle s_x]_x \equiv^1 ((\vec{a} \vec{b}) \cdot t')[x \mapsto \langle \vec{c}_x \rangle s_x]_x}$$

# NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle x_1 : n_1, \dots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \xrightarrow[u]{u} T_{\Sigma} V \xrightarrow{q_V} T_{\Sigma, \mathcal{A}} V$$

where  $V = \mathbb{A}^{\#n_1} + \dots + \mathbb{A}^{\#n_l}$

$$T_{\Sigma} V = \{ x_1 : n_1, \dots, x_l : n_l \vdash t \}$$

$$\begin{array}{c} \forall (\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} \\ \vdots \\ [\tilde{\Gamma}, T_{\Sigma} V] \# \mathbb{A}^{\#n} \xrightarrow[\llbracket t' \rrbracket]{\llbracket t \rrbracket} T_{\Sigma} V \xrightarrow{q_1} Y_1 \xrightarrow{q_2} Y_2 \cdots T_{\Sigma, \mathcal{A}} V \text{colim} \\ \vdots \end{array} \quad \begin{array}{c} F_{\Sigma} T_{\Sigma} V \xrightarrow{F_{\Sigma} q_1} F_{\Sigma} Y_1 \xrightarrow{F_{\Sigma} q_2} F_{\Sigma} Y_2 \cdots F_{\Sigma} T_{\Sigma, \mathcal{A}} V \\ \widehat{\mu}_V \downarrow \quad \text{po} \quad \downarrow \widehat{\sigma}_V \\ \swarrow \quad \searrow \\ \xrightarrow{q_1} Y_1 \xrightarrow{q_2} Y_2 \cdots \xrightarrow{q_V} \end{array}$$

$$\text{Ref}^2 \frac{}{t \equiv^2 t} \quad \text{Sym}^2 \frac{t \equiv^2 t'}{t' \equiv^2 t} \quad \text{Trans}^2 \frac{t \equiv^2 t' \quad t' \equiv^2 t''}{t \equiv^2 t''}$$

$$\text{Cong}^2 \frac{t_i \equiv^1 t'_i \quad (1 \leq i \leq k)}{\circ t_1 \dots t_k \equiv^2 \circ t'_1 \dots t'_k} \quad \circ \in \Sigma(k) \quad \text{Inc}^2 \frac{t \equiv^1 t'}{t \equiv^2 t'}$$

# NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle x_1 : n_1, \dots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \xrightarrow[u]{u} T_{\Sigma} V \xrightarrow{q_V} T_{\Sigma \mathcal{A}} V$$

where  $V = \mathbb{A}^{\#n_1} + \dots + \mathbb{A}^{\#n_l}$

$$T_{\Sigma} V = \{ x_1 : n_1, \dots, x_l : n_l \vdash t \}$$

$$\begin{array}{c} \forall (\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} \\ \vdots \\ [\tilde{\Gamma}, T_{\Sigma} V] \# \mathbb{A}^{\#n} \xrightarrow[\llbracket t' \rrbracket]{\llbracket t \rrbracket} T_{\Sigma} V \end{array} \begin{array}{c} \xrightarrow{F_{\Sigma} q_1} F_{\Sigma} Y_1 \xrightarrow{F_{\Sigma} q_2} F_{\Sigma} Y_2 \cdots F_{\Sigma} T_{\Sigma \mathcal{A}} V \\ \widehat{\mu}_V \downarrow \quad \text{po} \quad \downarrow \widehat{\sigma}_V \\ \xrightarrow{q_1} Y_1 \xrightarrow{q_2} Y_2 \cdots T_{\Sigma \mathcal{A}} V^{\text{colim}} \\ \xrightarrow{q_V} \end{array}$$

$$\text{Ref}^n \frac{}{t \equiv^n t} \quad \text{Sym}^n \frac{t \equiv^n t'}{t' \equiv^n t} \quad \text{Trans}^n \frac{t \equiv^n t' \quad t' \equiv^n t''}{t \equiv^n t''}$$

$$\text{Cong}^n \frac{t_i \equiv^{n-1} t'_i \quad (1 \leq i \leq k)}{o t_1 \dots t_k \equiv^n o t'_1 \dots t'_k} \quad o \in \Sigma(k) \quad \text{Inc}^n \frac{t \equiv^{n-1} t'}{t \equiv^n t'}$$



# NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle x_1 : n_1, \dots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \xrightarrow[u]{u} T_\Sigma V \xrightarrow{q_V} T_{\Sigma, \mathcal{A}} V$$

where  $V = \mathbb{A}^{\#n_1} + \dots + \mathbb{A}^{\#n_l}$

$$T_\Sigma V = \{ x_1 : n_1, \dots, x_l : n_l \vdash t \}$$

$$\begin{array}{c} \forall (\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} \\ \vdots \\ [\tilde{\Gamma}, T_\Sigma V] \# \mathbb{A}^{\#n} \xrightarrow[\llbracket t' \rrbracket]{\llbracket t \rrbracket} T_\Sigma V \xrightarrow{q_1} Y_1 \xrightarrow{q_2} Y_2 \cdots T_{\Sigma, \mathcal{A}} V_{\text{colim}} \\ \vdots \end{array} \quad \begin{array}{c} F_\Sigma T_\Sigma V \xrightarrow{F_\Sigma q_1} F_\Sigma Y_1 \xrightarrow{F_\Sigma q_2} F_\Sigma Y_2 \cdots F_\Sigma T_{\Sigma, \mathcal{A}} V \\ \widehat{\mu_V} \downarrow \quad \text{po} \quad \downarrow \widehat{\sigma_V} \\ T_\Sigma V \xrightarrow{q_1} Y_1 \xrightarrow{q_2} Y_2 \cdots T_{\Sigma, \mathcal{A}} V_{\text{colim}} \\ \xrightarrow{q_V} \end{array}$$

$$\text{Ref}^\omega \frac{}{t \equiv^\omega t} \quad \text{Sym}^\omega \frac{t \equiv^\omega t'}{t' \equiv^\omega t} \quad \text{Trans}^\omega \frac{t \equiv^\omega t' \quad t' \equiv^\omega t''}{t \equiv^\omega t''}$$

$$\text{Axiom}^\omega \frac{(\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} \quad \vec{b}^n \# \{ \langle \vec{c}_x \rangle s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_\Sigma V] \}_{x \in |\Gamma|}}{((\vec{a} \vec{b}) \cdot t)[x \mapsto \langle \vec{c}_x \rangle s_x]_x \equiv^\omega ((\vec{a} \vec{b}) \cdot t')[x \mapsto \langle \vec{c}_x \rangle s_x]_x}$$

$$\text{Cong}^\omega \frac{t_i \equiv^\omega t'_i \quad (1 \leq i \leq k)}{\circ t_1 \dots t_k \equiv^\omega \circ t'_1 \dots t'_k} \quad \circ \in \Sigma(k)$$

# NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle x_1 : n_1, \dots, x_l : n_l \vdash u \equiv v$$



$$\mathbb{A}^{\#m} \xrightarrow[u]{u} T_{\Sigma} V \xrightarrow{q_V} T_{\Sigma, \mathcal{A}} V$$



$$u \equiv^{\omega} v$$

$$\text{Ref}^{\omega} \frac{}{t \equiv^{\omega} t} \quad \text{Sym}^{\omega} \frac{t \equiv^{\omega} t'}{t' \equiv^{\omega} t} \quad \text{Trans}^{\omega} \frac{t \equiv^{\omega} t' \quad t' \equiv^{\omega} t''}{t \equiv^{\omega} t''}$$

$$\text{Axiom}^{\omega} \frac{(\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} \quad \vec{b}^n \# \{ \langle \vec{c}_x \rangle s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_{\Sigma} V] \}_{x \in |\Gamma|}}{((\vec{a} \vec{b}) \cdot t)[x \mapsto \langle \vec{c}_x \rangle s_x]_x \equiv^{\omega} ((\vec{a} \vec{b}) \cdot t')[x \mapsto \langle \vec{c}_x \rangle s_x]_x}$$

$$\text{Cong}^{\omega} \frac{t_i \equiv^{\omega} t'_i \quad (1 \leq i \leq k)}{\circ t_1 \dots t_k \equiv^{\omega} \circ t'_1 \dots t'_k} \quad \circ \in \Sigma(k)$$

# NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle x_1 : n_1, \dots, x_l : n_l \vdash u \equiv v$$



$$\mathbb{A}^{\#m} \xrightarrow[u]{u} T_{\Sigma} V \xrightarrow{q_V} T_{\Sigma, \mathcal{A}} V$$



$$u \equiv^{\omega} v$$

$\Updownarrow$  standard term rewriting

$$u \xleftarrow{\text{Axiom}^{\omega}; (\text{Cong}^{\omega})^*} t_1 \longleftrightarrow \dots \longleftrightarrow t_{n-1} \xleftarrow{\text{Axiom}^{\omega}; (\text{Cong}^{\omega})^*} v$$

$$\text{Ref}^{\omega} \frac{}{t \equiv^{\omega} t} \quad \text{Sym}^{\omega} \frac{t \equiv^{\omega} t'}{t' \equiv^{\omega} t} \quad \text{Trans}^{\omega} \frac{t \equiv^{\omega} t' \quad t' \equiv^{\omega} t''}{t \equiv^{\omega} t''}$$

$$\text{Axiom}^{\omega} \frac{(\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} \quad \vec{b}^n \# \{ \langle \vec{c}_x \rangle s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_{\Sigma} V] \}_{x \in |\Gamma|}}{((\vec{a} \vec{b}) \cdot t)[x \mapsto \langle \vec{c}_x \rangle s_x]_x \equiv^{\omega} ((\vec{a} \vec{b}) \cdot t')[x \mapsto \langle \vec{c}_x \rangle s_x]_x}$$

$$\text{Cong}^{\omega} \frac{t_i \equiv^{\omega} t'_i \quad (1 \leq i \leq k)}{o t_1 \dots t_k \equiv^{\omega} o t'_1 \dots t'_k} \quad o \in \Sigma(k)$$

# NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle x_1 : n_1, \dots, x_l : n_l \vdash u \equiv v$$



$$\Delta \#_m \xrightarrow{u} T_{\Sigma} V \quad q_V : T_{\Sigma} V$$

As proofs of  $u \equiv^{\omega} v$  can be turned into proofs in NTEL,  
NTEL is complete.

$$u \xleftarrow{\text{Axiom}^{\omega}; (\text{Cong}^{\omega})} t_1 \longleftrightarrow \dots \longleftrightarrow t_{n-1} \xleftarrow{\text{Axiom}^{\omega}; (\text{Cong}^{\omega})} v$$

$$\text{Ref}^{\omega} \frac{}{t \equiv^{\omega} t} \quad \text{Sym}^{\omega} \frac{t \equiv^{\omega} t'}{t' \equiv^{\omega} t} \quad \text{Trans}^{\omega} \frac{t \equiv^{\omega} t' \quad t' \equiv^{\omega} t''}{t \equiv^{\omega} t''}$$

$$\text{Axiom}^{\omega} \frac{(\langle \vec{a}^n \rangle \Gamma \vdash t \equiv t') \in \mathcal{A} \quad \vec{b}^n \# \{ \langle \vec{c}_x \rangle s_x \in [\mathbb{A}^{\# \Gamma(x)}, T_{\Sigma} V] \}_{x \in |\Gamma|}}{((\vec{a} \vec{b}) \cdot t)[x \mapsto \langle \vec{c}_x \rangle s_x]_x \equiv^{\omega} ((\vec{a} \vec{b}) \cdot t')[x \mapsto \langle \vec{c}_x \rangle s_x]_x}$$

$$\text{Cong}^{\omega} \frac{t_i \equiv^{\omega} t'_i \quad (1 \leq i \leq k)}{\circ t_1 \dots t_k \equiv^{\omega} \circ t'_1 \dots t'_k} \quad \circ \in \Sigma(k)$$

The **NTEL** is equivalent to the **Nominal Equational Logic** (Gabbay & Mathijssen 06; Clouston & Pitts 07).

$$\begin{aligned} \langle a, b \rangle x : 1 \vdash L_a x(a) \equiv L_b x(b) &\cong b \# x \vdash L_a x \equiv L_b (a b)x \\ \langle a \rangle x : 0 \vdash L_a (A x() V_a) \equiv x() &\cong a \# x \vdash L_a (A x V_a) \equiv x \end{aligned}$$

- ① Equational logic
  - algebraic theories and first-order equational logic  
(*Set*)
  - Nominal Equational Logic  
(Gabbay & Mathijssen 06; Clouston & Pitts 07)  
(*Nom*)
  - higher-order equational logic
- ② Rewriting system/logic
  - first-order Term Rewriting System (*Pre*)
  - Binding Term Rewriting System (Hamana 03) (*Pre<sup>II</sup>*)
  - Nominal Rewriting System (Fernández, Gabbay, Mackie 04)  
(*Perm// $\mathcal{P}(\mathbb{A})$* )  
cf. NTERS (*PreNom*)
  - Combinatory Reduction System (Klop 80) (*Pre<sup>IF</sup>*)
  - Higher-order Rewrite System (Nipkow 91)
  - Maude system (Meseguer *et al.*)