

Term Equational Systems and Logics

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Outline of the talk

- 1 **Definition** of Term Equational System (TES)
- 2 **Theory** of Term Equational Systems & Logics
- 3 **Applications**

Overview

Overview: Definition of TES

Term Equational Systems are a framework for developing equational logics.

Idea of Term Equational System

Abstract Syntax



\implies

- Terms in context
- Equational judgements
- Algebraic model theory

Universe of discourse

1 Term Equational Logic (TEL)

- Deductive system
- Sound (not necessarily complete)

2 Internal completeness & Construction of free models

- Reasoning by rewriting
- Sound and Complete

Overview: Applications

- 1 (Multi-sorted) First-order equational logic [treated in the talk]
- 2 (Multi-sorted) Nominal equational logic [treated in the paper]
(Gabbay & Mathijssen 06; Clouston & Pitts 07)
- 3 (Multi-sorted) Binding equational logic
(Hamana 03)
- 4 (Multi-sorted) Second-order equational logic
in the context of second-order abstract syntax (Fiore 08)
(cf. Equational fragment of Combinatory Reduction System
(Klop 80))

Definition

Definition: I. Enriched universe and Monad

- **Enriched universe** $\mathcal{U} = (\mathcal{V}, \underline{\mathcal{C}})$
 - \mathcal{V} : a symmetric monoidal closed category
 - $\underline{\mathcal{C}}$: a \mathcal{V} -category with tensors \otimes and powers $[-, =]$
 $- \otimes = : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$ $[-, =] : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$
- **\mathcal{V} -monad** \mathbf{T} on $\underline{\mathcal{C}}$
 \cong monad $\mathbf{T} = (T, \eta, \mu)$ on \mathcal{C} with strength τ
 $\tau_{V, X} : V \otimes TX \rightarrow T(V \otimes X)$

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Main example of enriched universe

Symmetric monoidal closed category (enriched over itself)

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Example (Multi-sorted First-order Equational Logic)

For S a set of sorts, Σ an S -sorted signature,

- $\mathcal{U}_\Sigma = (\mathbf{Set}, \mathbf{Set}^S)$
 $V \otimes \{X_s\}_{s \in S} = \{V \times X_s\}_{s \in S}$
 $[V, \{X_s\}_{s \in S}] = \{[V, X_s]\}_{s \in S}$
- \mathbf{T}_Σ with strength $\bar{\tau} =$ free monad on the endofunctor F_Σ with strength $\tau_{V, X} : V \otimes F_\Sigma X \rightarrow F_\Sigma(V \otimes X)$
 $F_\Sigma(\{X_s\}_{s \in S}) = \left\{ \coprod_{f: s_1 \dots s_n \rightarrow s \in \Sigma} X_{s_1} \times \dots \times X_{s_n} \right\}_{s \in S}$
 $\tau_{V, X}(v, f(x_1, \dots, x_n)) = f((v, x_1), \dots, (v, x_n))$

Definition: II. Terms and Judgements

Definition (Terms and Judgements)

Given $\mathcal{U} = (\mathcal{V}, \underline{\mathcal{C}})$ and $\mathbf{T} = (T, \eta, \mu, \tau)$,

- 1 **Generalised Term** $A, C \vdash t$ as $t : A \rightarrow TC \in \mathcal{C}$
- 2 **Generalised Equation** $A, C \vdash t \equiv t'$

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Example (continued)

Syntactic terms \cong Generalised terms

$$\{t \mid \Gamma \vdash t : s\} \cong T_{\Sigma} \tilde{\Gamma}(s) \cong \mathbf{Set}^S(\tilde{s}, T_{\Sigma} \tilde{\Gamma})$$

where

$$\begin{aligned} \tilde{s}(r) &= \begin{cases} 1 & \text{if } r = s \\ \emptyset & \text{otherwise} \end{cases} \\ \tilde{\Gamma}(r) &= \{x \mid x : r \in \Gamma\} \end{aligned}$$

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Object Variables

Meta Variables

in higher order examples

Example (continued)

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Definition: III. Model Theory

Definition (Models and Satisfaction relation)

Given $\mathcal{U} = (\mathcal{V}, \underline{\mathcal{C}})$ and $\mathbf{T} = (T, \eta, \mu, \tau)$,

- Models for \mathbf{T} : **Eilenberg-Moore algebras** for \mathbf{T}

$$(X, \xi : TX \rightarrow X)$$

- Satisfaction relation

$$(X, \xi) \models A, C \vdash t \equiv t' \iff \underline{\mathcal{C}}(C, X) \otimes A \begin{array}{c} \xrightarrow{[[t]]_{(X, \xi)}} \\ \parallel \\ \xrightarrow{[[t']]_{(X, \xi)}} \end{array} X$$

where $[[t]]_{(X, \xi)}$ is the transpose of

$$\underline{\mathcal{C}}(C, X) \xrightarrow{\mathbf{T}} \underline{\mathcal{C}}(\mathbf{T}C, \mathbf{T}X) \xrightarrow{\underline{\mathcal{C}}(t, \xi)} \underline{\mathcal{C}}(A, X)$$

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Example (continued)

- Models for \mathbf{T}_Σ : F_Σ -algebras $(X, \xi : F_\Sigma X \rightarrow X)$

$$\forall f: s_1 \dots s_n \rightarrow s \in \Sigma \quad \xi_f : X(s_1) \times \dots \times X(s_n) \rightarrow X(s)$$

- For $(\Gamma \vdash t \equiv t' : s)$,

$$[[t]] : \prod_{r \in S} X(r) \{x \mid x : r \in \Gamma\} \rightarrow X(s)$$

Definition: IV. Term Equational System

Definition (Term Equational System)

- A TES $\mathbb{S} = (\mathcal{V}, \underline{\mathcal{L}}, \underline{\mathbf{T}}, \mathcal{A})$ is given by
 - a enriched universe $(\mathcal{V}, \underline{\mathcal{L}})$,
 - a \mathcal{V} -monad $\underline{\mathbf{T}}$, and
 - a set \mathcal{A} of generalised equations.
- Models for \mathbb{S} (\mathbb{S} -algebras): Models for $\underline{\mathbf{T}}$ satisfying \mathcal{A}
- $\mathbb{S}\text{-Alg}$: The category of \mathbb{S} -algebras.

Theory I.

Term Equational Logic

Theory: Term Equational Logic

- Equivalence relation

$$\frac{}{A, C \vdash t \equiv t} \quad \frac{A, C \vdash t \equiv t'}{A, C \vdash t' \equiv t} \quad \frac{A, C \vdash t \equiv t' \quad A, C \vdash t' \equiv t''}{A, C \vdash t \equiv t''}$$

- Axiom

$$\frac{}{A, C \vdash t \equiv t'} (A, C \vdash t \equiv t') \in \mathcal{A}$$

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Example: Multi-sorted First-order Equational Logic (MFoEL)

$$\frac{}{\Gamma \vdash t \equiv t' : s} \quad \frac{\Gamma \vdash t \equiv t' : s}{\Gamma \vdash t' \equiv t : s} \quad \frac{\Gamma \vdash t \equiv t' : s \quad \Gamma \vdash t' \equiv t'' : s}{\Gamma \vdash t \equiv t'' : s}$$

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Theory: Term Equational Logic

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- Axiom

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- Local Character

$$\frac{A_i, C \vdash t \circ e_i \equiv t' \circ e_i \quad (i \in I)}{A, C \vdash t \equiv t'} \{e_i : A_i \rightarrow A\}_{i \in I} \text{ jointly epi}$$

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- Substitution

$$\frac{A, B \vdash t \equiv t' \quad B, C \vdash s \equiv s'}{A, C \vdash t[s] \equiv t'[s']} \longrightarrow \boxed{A \xrightarrow{t} TB \xrightarrow{Ts} TTC \xrightarrow{\mu} TC}$$

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Subst \leftarrow Loc. Char. + Substitution

$$\frac{\{x_i : s_i\}_i \vdash t \equiv t' : t \quad \{\Gamma \vdash s_i \equiv s'_i : s_i\}_{i \in I}}{\Gamma \vdash t[x_i \mapsto s_i]_i \equiv t'[x_i \mapsto s'_i]_i : t} \text{Subst}$$

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- Tensor Extension

$$\frac{A, C \vdash t \equiv t'}{V \otimes A, V \otimes C \vdash \langle V \rangle t \equiv \langle V \rangle t'} \rightarrow V \otimes A \xrightarrow{id \otimes t} V \otimes TC \xrightarrow{\tau} T(V \otimes C)$$

Theorem (Soundness of TEL)

For a TES $\mathbb{S} = (\mathcal{V}, \underline{\mathcal{C}}, \underline{\mathbf{T}}, \mathcal{A})$

$A, C \vdash t \equiv t'$ is derivable from \mathcal{A}



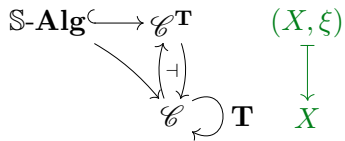
$\forall (X, \xi) \in \mathbb{S}\text{-Alg} \quad (X, \xi) \models A, C \vdash t \equiv t'$

Theory II.

Internal Completeness & Free Construction

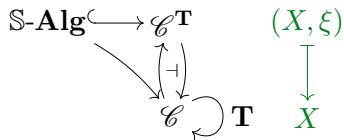
Theory: Internal Completeness

For a TES $\mathbb{S} = (\mathcal{V}, \underline{\mathcal{C}}, \underline{\mathbf{T}}, \mathcal{A})$



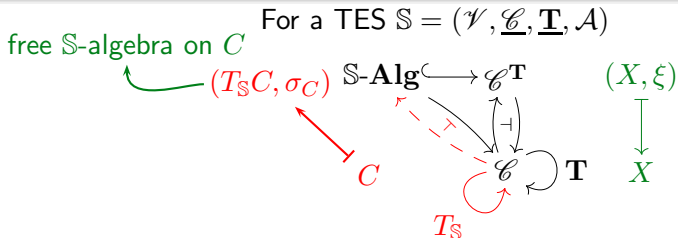
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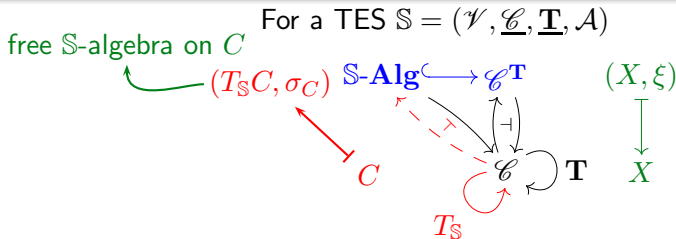
Theory: Internal Completeness



$$\forall (X, \xi) \in \mathbb{S}\text{-Alg} \quad (X, \xi) \models A, C \vdash t \equiv t'$$

$$\iff (T_{\mathbb{S}}C, \sigma_C) \models A, C \vdash t \equiv t'$$

Theory: Internal Completeness



$$\forall (X, \xi) \in \mathbb{S}\text{-Alg} \quad (X, \xi) \models A, C \vdash t \equiv t'$$



$$(T_{\mathbb{S}}C, \sigma_C) \models A, C \vdash t \equiv t'$$



$$q_C \circ t = q_C \circ t' : A \xrightarrow[t']{t} TC \xrightarrow{q_C} T_{\mathbb{S}}C$$

where the quotient map q_C is given by

$$\begin{array}{ccc} TTC & \xrightarrow{Tq_C} & TT_{\mathbb{S}}C \\ \mu_C \downarrow & & \downarrow \sigma_C \\ TC & \xrightarrow{\exists! q_C} & T_{\mathbb{S}}C \\ \eta_C \uparrow & \nearrow \eta_{\mathbb{S}C} & \\ C & & \end{array}$$

Theory: Construction of free models

For a TES $\mathbb{S} = (\mathcal{V}, \underline{\mathcal{L}}, \underline{\mathbf{T}}, \mathcal{A})$

The theory of **Equational System** (Fiore & Hur, ICALP 07) provides an **explicit construction** of **free \mathbb{S} -algebras** and the quotient map $q : T \twoheadrightarrow T_{\mathbb{S}}$.

Theory: Construction of free models

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If

- \mathcal{C} is cocomplete.
- T preserves colimits of ω -chains and epimorphisms.
- For every $(B, D \vdash s \equiv s') \in \mathcal{A}$,
 $\underline{\mathcal{C}}(D, -) \otimes B$ preserves colimits of ω -chains and epimorphisms.

Theory: Construction of free models

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Then $(T_{\mathbb{S}}C, \sigma_C)$ and $q_C : TC \rightarrow T_{\mathbb{S}}C$ can be constructed as follows:

$$\begin{array}{ccccccc}
 \forall (B, D \vdash s \equiv s') \in \mathcal{A} & & & & & & \\
 \vdots & & & & & & \\
 \underline{\mathcal{C}}(D, TC) \otimes B & \xrightarrow{\llbracket s \rrbracket_{(TC, \mu_C)}} & TTC & \xrightarrow{Tq_1} & TY_1 & \xrightarrow{Tq_2} & TY_2 & \longrightarrow & \cdots & \longrightarrow & TT_{\mathbb{S}}C \\
 & & \mu_C \downarrow & \searrow & \text{po} & \searrow & \text{po} & \searrow & & & \downarrow \sigma_C \\
 & & & & TC & \xrightarrow{q_1} & Y_1 & \xrightarrow{q_2} & Y_2 & \longrightarrow & \cdots & \longrightarrow & T_{\mathbb{S}}C \\
 & & & & \llbracket s' \rrbracket_{(TC, \mu_C)} & & & & & & & & \downarrow q_C \\
 \vdots & & & & & & & & & & & & \text{colim}
 \end{array}$$

Theory: Construction of free models

For a TES $\mathbb{S} = (\mathcal{V}, \underline{\mathcal{C}}, \mathbf{T}, \mathcal{A})$

The theory of **Equational System** (Fiore & Hur, ICALP 07) provides an **explicit construction** of **free \mathbb{S} -algebras** and the quotient map $q : T \rightarrow T_{\mathbb{S}}$.

- If
- \mathcal{C} is cocomplete.
 - Σ preserves colimits of ω -chains and epimorphisms.
 - For every $(B, D \vdash s \equiv s') \in \mathcal{A}$,
 $\underline{\mathcal{C}}(D, -) \otimes B$ preserves colimits of ω -chains and epimorphisms.
 - \mathbf{T} is a free monad of an endofunctor Σ .

Then $(T_{\mathbb{S}}C, \sigma_C)$ and $q_C : TC \rightarrow T_{\mathbb{S}}C$ can be constructed as follows:

$$\begin{array}{ccccccc}
 \forall (B, D \vdash s \equiv s') \in \mathcal{A} & & \Sigma TC & \xrightarrow{\Sigma q_1} & \Sigma Y_1 & \xrightarrow{\Sigma q_2} & \Sigma Y_2 & \longrightarrow & \cdots & \longrightarrow & \Sigma T_{\mathbb{S}}C \\
 \vdots & & \downarrow \widehat{\mu}_C & \searrow & \text{po} & \searrow & \text{po} & \searrow & & & \downarrow \widehat{\sigma}_C \\
 \underline{\mathcal{C}}(D, TC) \otimes B & \xrightarrow{[s]_{(TC, \mu_C)}} & TC & \xrightarrow{q_1} & Y_1 & \xrightarrow{q_2} & Y_2 & \longrightarrow & \cdots & \longrightarrow & T_{\mathbb{S}}C \\
 \vdots & & \downarrow [s']_{(TC, \mu_C)} & \searrow & & \searrow & & \searrow & & & \downarrow \text{colim} \\
 & & & & & & & & & & &
 \end{array}$$

Internal Completeness

$$\begin{array}{c} \mathbb{S}\text{-Alg} \models A, C \vdash t \equiv t' \\ \Downarrow \\ q_C \circ t = q_C \circ t' : A \begin{array}{c} \xrightarrow{t} \\ \xrightarrow{t'} \end{array} TC \xrightarrow{q_C} T_{\mathbb{S}}C \end{array}$$

+

Explicit construction of $T_{\mathbb{S}}C$ and q_C

↓

We may synthesise a sound and complete logic.

Theory: Towards Completeness

Example (continued)

$$\mathbb{S}\text{-Alg} \models \nabla \vdash u \equiv v : r \quad \text{Int Comp} \quad \tilde{r} \xrightarrow[u]{u} T_{\Sigma} \tilde{\nabla} \xrightarrow{q_{\tilde{\nabla}}} T_{\Sigma \mathbb{S}} \tilde{\nabla}$$

Theory: Towards Completeness

Example (continued)

$$\mathbb{S}\text{-Alg} \models \nabla \vdash u \equiv v : r \quad \text{Int Comp} \quad \tilde{r} \xrightarrow[u]{u} T_\Sigma \tilde{\nabla} \xrightarrow{q_{\tilde{\nabla}}} T_{\Sigma \mathbb{S}} \tilde{\nabla}$$

$$\begin{array}{c} \forall (\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \\ \vdots \\ \xrightarrow{\llbracket t \rrbracket} \\ \vdots \\ \xrightarrow{\llbracket t' \rrbracket} \end{array} \begin{array}{c} F_\Sigma T_\Sigma \tilde{\nabla} \\ \downarrow \widehat{\mu}_{\tilde{\nabla}} \\ T_\Sigma \tilde{\nabla} \end{array} \begin{array}{c} \xrightarrow{F_\Sigma q_1} \\ \searrow \\ \xrightarrow{q_1} \end{array} \begin{array}{c} F_\Sigma (T_\Sigma \tilde{\nabla} /_{\approx 1}) \\ \text{po} \\ T_\Sigma \tilde{\nabla} /_{\approx 1} \end{array} \begin{array}{c} \cdots \\ \searrow \\ \xrightarrow{q_2} \end{array} \begin{array}{c} F_\Sigma (T_\Sigma \tilde{\nabla} /_{\approx \omega}) \\ \downarrow \widehat{\sigma}_{\tilde{\nabla}} \\ T_\Sigma \tilde{\nabla} /_{\approx \omega} \end{array} \begin{array}{c} \xrightarrow{\text{colim}} \\ \xrightarrow{q_{\tilde{\nabla}}} \end{array}$$

Theory: Towards Completeness

Example (continued)

$$\mathbb{S}\text{-Alg} \models \nabla \vdash u \equiv v : r \quad \text{Int Comp} \quad \tilde{r} \xrightarrow{u} T_\Sigma \tilde{\nabla} \xrightarrow{q_{\tilde{\nabla}}} T_{\Sigma \mathbb{S}} \tilde{\nabla}$$

$$\begin{array}{c} \forall (\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \\ \vdots \\ \xrightarrow{\llbracket t \rrbracket} \\ \xrightarrow{\llbracket t' \rrbracket} \\ \vdots \end{array} \quad \begin{array}{c} F_\Sigma T_\Sigma \tilde{\nabla} \xrightarrow{F_\Sigma q_1} F_\Sigma (T_\Sigma \tilde{\nabla} / \approx_1) \quad \dots \quad F_\Sigma (T_\Sigma \tilde{\nabla} / \approx_\omega) \\ \downarrow \widehat{\mu}_{\tilde{\nabla}} \quad \searrow \text{po} \quad \searrow \quad \downarrow \widehat{\sigma}_{\tilde{\nabla}} \\ T_\Sigma \tilde{\nabla} \xrightarrow{q_1} T_\Sigma \tilde{\nabla} / \approx_1 \xrightarrow{q_2} T_\Sigma \tilde{\nabla} / \approx_2 \dots T_\Sigma \tilde{\nabla} / \approx_\omega \\ \xrightarrow{q_{\tilde{\nabla}}} \text{colim} \end{array}$$

\approx_s^1 on $T_\Sigma \tilde{\nabla}(s) = \{t \mid \nabla \vdash t : s\}$

$$\text{Ref}^1 \frac{}{t \approx_s^1 t} \quad \text{Sym}^1 \frac{t \approx_s^1 t'}{t' \approx_s^1 t} \quad \text{Trans}^1 \frac{t \approx_s^1 t' \quad t' \approx_s^1 t''}{t \approx_s^1 t''}$$

$$\text{Axiom}^1 \frac{(\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \quad \{\nabla \vdash s_i : s_i\}_{x_i : s_i \in \Gamma}}{t[x_i \mapsto s_i] \approx_s^1 t'[x_i \mapsto s_i]}$$

Theory: Towards Completeness

Example (continued)

$$\mathbb{S}\text{-Alg} \models \nabla \vdash u \equiv v : r \quad \text{Int Comp} \quad \tilde{r} \xrightarrow[u]{u} T_\Sigma \tilde{\nabla} \xrightarrow{q_{\tilde{\nabla}}} T_{\Sigma \mathbb{S}} \tilde{\nabla}$$

$$\begin{array}{c} \forall (\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \\ \vdots \\ \xrightarrow{\llbracket t \rrbracket} \\ \xrightarrow{\llbracket t' \rrbracket} \\ \vdots \end{array} \quad \begin{array}{c} F_\Sigma T_\Sigma \tilde{\nabla} \xrightarrow{F_\Sigma q_1} F_\Sigma (T_\Sigma \tilde{\nabla} /_{\approx 1}) \quad \dots \quad F_\Sigma (T_\Sigma \tilde{\nabla} /_{\approx \omega}) \\ \downarrow \widehat{\mu}_{\tilde{\nabla}} \quad \searrow \text{po} \quad \swarrow \widehat{\sigma}_{\tilde{\nabla}} \\ T_\Sigma \tilde{\nabla} \xrightarrow{q_1} T_\Sigma \tilde{\nabla} /_{\approx 1} \xrightarrow{q_2} T_\Sigma \tilde{\nabla} /_{\approx 2} \dots T_\Sigma \tilde{\nabla} /_{\approx \omega} \\ \xrightarrow{q_{\tilde{\nabla}}} \text{colim} \end{array}$$

\approx_s^2 on $T_\Sigma \tilde{\nabla}(s) = \{t \mid \nabla \vdash t : s\}$

$$\text{Ref}^2 \frac{}{t \approx_s^2 t} \quad \text{Sym}^2 \frac{t \approx_s^2 t'}{t' \approx_s^2 t} \quad \text{Trans}^2 \frac{t \approx_s^2 t' \quad t' \approx_s^2 t''}{t \approx_s^2 t''}$$

$$\text{Cong}^2 \frac{t_i \approx_{s_i}^1 t'_i \quad (1 \leq i \leq k)}{f(t_1..t_k) \approx_s^2 f(t'_1..t'_k)} \quad f : s_1..s_k \rightarrow s \in \Sigma \quad \text{Ext}^2 \frac{t \approx_s^1 t'}{t \approx_s^2 t'}$$

Theory: Towards Completeness

Example (continued)

$$\mathbb{S}\text{-Alg} \models \nabla \vdash u \equiv v : r \quad \text{Int Comp} \quad \tilde{r} \xrightarrow[u]{u} T_\Sigma \tilde{\nabla} \xrightarrow{q_{\tilde{\nabla}}} T_{\Sigma \mathbb{S}} \tilde{\nabla}$$

$$\begin{array}{c} \forall (\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \\ \vdots \\ \xrightarrow{[t]} \\ \xrightarrow{[t']} \\ \vdots \end{array} \quad \begin{array}{c} F_\Sigma T_\Sigma \tilde{\nabla} \xrightarrow{F_\Sigma q_1} F_\Sigma (T_\Sigma \tilde{\nabla} /_{\approx 1}) \quad \dots \quad F_\Sigma (T_\Sigma \tilde{\nabla} /_{\approx \omega}) \\ \downarrow \widehat{\mu}_{\tilde{\nabla}} \quad \searrow \text{po} \quad \searrow \quad \downarrow \widehat{\sigma}_{\tilde{\nabla}} \\ T_\Sigma \tilde{\nabla} \xrightarrow{q_1} T_\Sigma \tilde{\nabla} /_{\approx 1} \xrightarrow{q_2} T_\Sigma \tilde{\nabla} /_{\approx 2} \quad \dots \quad T_\Sigma \tilde{\nabla} /_{\approx \omega} \\ \xrightarrow{q_{\tilde{\nabla}}} \text{colim} \end{array}$$

\approx_s^n on $T_\Sigma \tilde{\nabla}(s) = \{t \mid \nabla \vdash t : s\}$

$$\text{Ref}^n \frac{}{t \approx_s^n t} \quad \text{Sym}^n \frac{t \approx_s^n t'}{t' \approx_s^n t} \quad \text{Trans}^n \frac{t \approx_s^n t' \quad t' \approx_s^n t''}{t \approx_s^n t''}$$

$$\text{Cong}^n \frac{t_i \approx_{s_i}^{n-1} t'_i \quad (1 \leq i \leq k)}{f(t_1 \dots t_k) \approx_s^n f(t'_1 \dots t'_k)} \quad f : s_1 \dots s_k \rightarrow s \in \Sigma \quad \text{Ext}^n \frac{t \approx_s^{n-1} t'}{t \approx_s^n t'}$$

Theory: Towards Completeness

Example (continued)

$$\mathbb{S}\text{-Alg} \models \nabla \vdash u \equiv v : r \quad \text{Int Comp} \quad \Leftrightarrow \quad \tilde{r} \xrightarrow[u]{u} T_\Sigma \tilde{\nabla} \xrightarrow{q_{\tilde{\nabla}}} T_{\Sigma \mathbb{S}} \tilde{\nabla}$$

$$\begin{array}{c} \forall (\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \\ \vdots \\ \xrightarrow{\llbracket t \rrbracket} \\ \vdots \\ \xrightarrow{\llbracket t' \rrbracket} \end{array} \quad \begin{array}{c} F_\Sigma T_\Sigma \tilde{\nabla} \\ \downarrow \widehat{\mu}_{\tilde{\nabla}} \\ T_\Sigma \tilde{\nabla} \end{array} \xrightarrow{F_\Sigma q_1} F_\Sigma (T_\Sigma \tilde{\nabla} / \approx_1) \quad \dots \quad F_\Sigma (T_\Sigma \tilde{\nabla} / \approx_\omega) \\ \searrow \text{po} \quad \searrow \widehat{\sigma}_{\tilde{\nabla}} \\ T_\Sigma \tilde{\nabla} / \approx_1 \xrightarrow{q_2} T_\Sigma \tilde{\nabla} / \approx_2 \quad \dots \quad T_\Sigma \tilde{\nabla} / \approx_\omega \\ \xrightarrow{q_{\tilde{\nabla}}} \text{colim}$$

\approx_s^ω on $T_\Sigma \tilde{\nabla}(s) = \{t \mid \nabla \vdash t : s\}$

$$\text{Ref}^\omega \frac{}{t \approx_s^\omega t} \quad \text{Sym}^\omega \frac{t \approx_s^\omega t'}{t' \approx_s^\omega t} \quad \text{Trans}^\omega \frac{t \approx_s^\omega t' \quad t' \approx_s^\omega t''}{t \approx_s^\omega t''}$$

$$\text{Axiom}^\omega \frac{(\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \quad \{\nabla \vdash s_i : s_i\}_{x_i : s_i \in \Gamma}}{t[x_i \mapsto s_i] \approx_s^\omega t'[x_i \mapsto s_i]}$$

$$\text{Cong}^\omega \frac{t_i \approx_{s_i}^\omega t'_i \quad (1 \leq i \leq k)}{f(t_1..t_k) \approx_s^\omega f(t'_1..t'_k)} \quad f : s_1..s_k \rightarrow s \in \Sigma$$

Theory: Towards Completeness

$$\mathbb{S}\text{-Alg} \models \nabla \vdash u \equiv v : r$$

$$\begin{array}{c} \Downarrow \\ \tilde{r} \xrightarrow[u]{u} T_{\Sigma} \tilde{\nabla} \xrightarrow{q_{\tilde{\nabla}}} T_{\Sigma} \tilde{\nabla} / \approx^{\omega} \end{array}$$

$$\Downarrow$$

$$u \approx_r^{\omega} v$$

Term Rewriting

$$\Downarrow$$

$$u \xleftarrow{\tilde{1}_r} t_1 \xleftarrow{\tilde{1}_r} \dots \xleftarrow{\tilde{1}_r} t_{n-1} \xleftarrow{\tilde{1}_r} v$$

$$\text{Ref}^{\omega} \frac{}{t \approx_s^{\omega} t} \quad \text{Sym}^{\omega} \frac{t \approx_s^{\omega} t'}{t' \approx_s^{\omega} t} \quad \text{Trans}^{\omega} \frac{t \approx_s^{\omega} t' \quad t' \approx_s^{\omega} t''}{t \approx_s^{\omega} t''}$$

$$\text{Axiom}^{\omega} \frac{(\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \quad \{\nabla \vdash s_i : s_i\}_{x_i : s_i \in \Gamma}}{t[x_i \mapsto s_i] \approx_s^{\omega} t'[x_i \mapsto s_i]}$$

$$\text{Cong}^{\omega} \frac{t_i \approx_{s_i}^{\omega} t'_i \quad (1 \leq i \leq k)}{f(t_1..t_k) \approx_s^{\omega} f(t'_1..t'_k)} \quad f : s_1..s_k \rightarrow s \in \Sigma$$

Theory: Towards Completeness

$$\mathbb{S}\text{-Alg} \models \nabla \vdash u \equiv v : r$$



Completeness

As proofs of $u \approx_r^\omega v$ can be turned into proofs in MFoEL,
MFoEL is complete.

$$u \xleftrightarrow{\approx_r} t_1 \xleftrightarrow{\approx_r} \dots \xleftrightarrow{\approx_r} t_{n-1} \xleftrightarrow{\approx_r} v$$

$$\text{Ref}^\omega \frac{}{t \approx_s^\omega t} \quad \text{Sym}^\omega \frac{t \approx_s^\omega t'}{t' \approx_s^\omega t} \quad \text{Trans}^\omega \frac{t \approx_s^\omega t' \quad t' \approx_s^\omega t''}{t \approx_s^\omega t''}$$

$$\text{Axiom}^\omega \frac{(\Gamma \vdash t \equiv t' : s) \in \mathcal{A} \quad \{\nabla \vdash s_i : s_i\}_{x_i : s_i \in \Gamma}}{t[x_i \mapsto s_i] \approx_s^\omega t'[x_i \mapsto s_i]}$$

$$\text{Cong}^\omega \frac{t_i \approx_{s_i}^\omega t'_i \quad (1 \leq i \leq k)}{f(t_1..t_k) \approx_s^\omega f(t'_1..t'_k)} \quad f : s_1..s_k \rightarrow s \in \Sigma$$

Example (continued)

$$S = \{ \mathbf{F}, \mathbf{B} \}$$

$$\Sigma = \{ \text{True} : \mathbf{B}, \text{ False} : \mathbf{B}, \neg : \mathbf{B} \rightarrow \mathbf{B}, \\ \wedge : \mathbf{B}, \mathbf{B} \rightarrow \mathbf{B}, \vee : \mathbf{B}, \mathbf{B} \rightarrow \mathbf{B}, \\ \text{Foo} : \mathbf{F} \rightarrow \mathbf{B} \}$$

$$\mathcal{A} = \{ \vdash \neg \text{True} = \text{False} : \mathbf{B}, \quad \vdash \neg \text{False} = \text{True} : \mathbf{B}, \\ x : \mathbf{B} \vdash x \vee \neg x = \text{True} : \mathbf{B}, \quad x : \mathbf{B} \vdash x \wedge \neg x = \text{False} : \mathbf{B}, \\ x : \mathbf{B} \vdash x \vee x = x : \mathbf{B}, \quad x : \mathbf{B} \vdash x \wedge x = x : \mathbf{B}, \\ y : \mathbf{F} \vdash \text{Foo}(y) = \neg \text{Foo}(y) : \mathbf{B} \}$$

Subtlety of MFoEL [Goguen & Meseguer 1985]

Example (continued)

$$S = \{ \mathbf{F}, \mathbf{B} \}$$

$$\Sigma = \{ \text{True} : \mathbf{B}, \text{ False} : \mathbf{B}, \neg : \mathbf{B} \rightarrow \mathbf{B}, \\ \wedge : \mathbf{B}, \mathbf{B} \rightarrow \mathbf{B}, \vee : \mathbf{B}, \mathbf{B} \rightarrow \mathbf{B}, \\ \text{Foo} : \mathbf{F} \rightarrow \mathbf{B} \}$$

$$\mathcal{A} = \{ \vdash \neg \text{True} = \text{False} : \mathbf{B}, \vdash \neg \text{False} = \text{True} : \mathbf{B}, \\ x : \mathbf{B} \vdash x \vee \neg x = \text{True} : \mathbf{B}, x : \mathbf{B} \vdash x \wedge \neg x = \text{False} : \mathbf{B}, \\ x : \mathbf{B} \vdash x \vee x = x : \mathbf{B}, x : \mathbf{B} \vdash x \wedge x = x : \mathbf{B}, \\ y : \mathbf{F} \vdash \text{Foo}(y) = \neg \text{Foo}(y) : \mathbf{B} \}$$

False reasoning

Theorem: $\vdash \text{True} = \text{False} : \mathbf{B}$

Proof:

$$\begin{aligned} \text{True} &= \text{Foo}(y) \vee \neg \text{Foo}(y) = \text{Foo}(y) \vee \text{Foo}(y) = \text{Foo}(y) \\ &= \text{Foo}(y) \wedge \text{Foo}(y) = \text{Foo}(y) \wedge \neg \text{Foo}(y) = \text{False} \end{aligned}$$

Subtlety of MFoEL [Goguen & Meseguer 1985]

Example (continued)

$$S = \{ \mathbf{F}, \mathbf{B} \}$$

$$\Sigma = \{ \text{True} : \mathbf{B}, \text{ False} : \mathbf{B}, \neg : \mathbf{B} \rightarrow \mathbf{B},$$

$$\wedge : \mathbf{B}, \mathbf{B} \rightarrow \mathbf{B}, \vee : \mathbf{B}, \mathbf{B} \rightarrow \mathbf{B},$$

$$\mathbf{F} : \mathbf{B}, \mathbf{B} \}$$

$$\mathcal{A} = \{ \vdash$$

$$x$$

$$x$$

$$y$$

$$: \mathbf{F} \vdash \text{True} = \text{False} : \mathbf{B} \}$$

For a judgement $\nabla \vdash u \equiv v : r$

\approx_r^ω is a relation on the set $\{t \mid \nabla \vdash t : r\}$

se : \mathbf{B} ,

False reasoning

Theorem: $\vdash \text{True} = \text{False} : \mathbf{B}$

Proof:

$$\text{True} = \text{Foo}(y) \vee \neg \text{Foo}(y) = \text{Foo}(y) \vee \text{Foo}(y) = \text{Foo}(y)$$

$$= \text{Foo}(y) \wedge \text{Foo}(y) = \text{Foo}(y) \wedge \neg \text{Foo}(y) = \text{False}$$

Subtlety of MFoEL [Goguen & Meseguer 1985]

Example (continued)

$$S = \{ \mathbf{F}, \mathbf{B} \}$$

$$\Sigma = \{ \text{True} : \mathbf{B}, \text{ False} : \mathbf{B}, \neg : \mathbf{B} \rightarrow \mathbf{B},$$

$$\wedge : \mathbf{B}, \mathbf{B} \rightarrow \mathbf{B}, \vee : \mathbf{B}, \mathbf{B} \rightarrow \mathbf{B},$$

$$\mathbf{F} : \mathbf{F}, \mathbf{B} : \mathbf{B} \}$$

$$\mathcal{A} = \{ \vdash$$

x

x

y

$$: \mathbf{F} \vdash \text{True} = \text{False} : \mathbf{B}$$

For a judgement $\nabla \vdash u \equiv v : r$

\approx_r^ω is a relation on the set $\{t \mid \nabla \vdash t : r\}$

se : \mathbf{B} ,

Valid reasoning

Theorem: $y : \mathbf{F} \vdash \text{True} = \text{False} : \mathbf{B}$

Proof:

$$\text{True} = \text{Foo}(y) \vee \neg \text{Foo}(y) = \text{Foo}(y) \vee \text{Foo}(y) = \text{Foo}(y)$$

$$= \text{Foo}(y) \wedge \text{Foo}(y) = \text{Foo}(y) \wedge \neg \text{Foo}(y) = \text{False}$$

Applications

- Multi-sorted first-order equational logic ($\mathbf{Set}, \mathbf{Set}^S$)
- Nominal equational logic ($\mathbf{Nom}, \mathbf{Nom}$)
(Gabbay & Mathijssen 06; Clouston & Pitts 07)
- Binding equational logic ($\mathbf{Set}^{\mathbb{I}}, \mathbf{Set}^{\mathbb{I}}$)
(Hamana 03)
- Second-order equational logic ($\mathbf{Set}^{\mathbb{F}}, \mathbf{Set}^{\mathbb{F}}$)
in the context of second-order abstract syntax (Fiore 08)

Extension

- Term Equational Rewriting Systems and Logics
e.g. Rewriting logic (Meseguer et al.)