
ON THE MATHEMATICAL SYNTHESIS OF EQUATIONAL LOGICS*

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ABSTRACT. We provide a mathematical theory and methodology for synthesising equational logics from algebraic metatheories. We illustrate our methodology by means of two applications: a rational reconstruction of Birkhoff’s Equational Logic and a new equational logic for reasoning about algebraic structure with name-binding operators.

INTRODUCTION

Birkhoff (1935) initiated the general study of algebraic structure. Importantly for our concerns here, his development was from (universal) algebra to (equational) logic. Birkhoff’s starting point was the informal conception of algebra based on familiar concrete examples. Abstracting from these, he introduced the concepts of signature and equational presentation, and thereby formalised what is now our notion of (abstract) algebra. Subsequently he set up the model theory of equational presentations (varieties) and analysed their structure from the standpoint of logical inference for algebraic languages. In doing so, he introduced Equational Logic as a sound and complete deductive system for reasoning about equational assertions in Algebraic Theories.

Since Birkhoff’s work, our understanding of algebraic structure has deepened; having been both systematised and extended (see *e.g.* Lawvere (1963), Ehresmann (1968), Burroni (1981), Kelly and Power (1993), Power (1999)). On the other hand, the development of equational logics has remained ad hoc. The main aim of the current work is to fill in this gap.

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* This paper gives a complete development (with proofs) of results announced in Fiore and Hur (2008). However, for simplicity of exposition, these results are restricted here to the case of *mono-sorted* algebraic theories and equational logics.

Our standpoint is that equational logics should arise from algebraic structure. In this direction, our first purpose is to provide a mathematical theory and methodology for synthesising equational logics from algebraic metatheories (Part I). Our second purpose is to establish the practicality of the approach. In this respect, we illustrate our methodology by means of two applications: a rational reconstruction of Birkhoff’s Equational Logic and a new equational logic for reasoning about algebraic structure with name-binding operators (Part II).

PART I. THEORY

In this first part of the paper, we present our mathematical framework for synthesising equational logics. For simplicity of exposition, we restrict attention to the *mono-sorted* context. As such, we consider algebraic metatheories given by strong monads on symmetric monoidal categories. These provide algebraic structure that allows the specification of equational presentations in the form of *Monadic Equational Systems* (Section 1). Monadic Equational Systems come equipped with a canonical model theory whereby models are Eilenberg-Moore algebras satisfying the equations. An *Equational Metalogic* (Section 2) for reasoning about equality in such models is presented. This deductive system has been designed to guarantee sound derivations. As for completeness (Section 3), a mathematical justification of the well-known use of free constructions in equational completeness proofs is given, and this is backed up with an inductive method for constructing free algebras.

1. MONADIC EQUATIONAL SYSTEMS

Monadic Equational Systems (MESs) are defined and their model theory is explained.

1.1. Monadic Equational Systems. The concept of MES provides a general abstract notion of equational presentation.

Definition 1.1 (Terms and equations). A *term* for an endofunctor T on a category \mathcal{C} of arity A and coarity C is a Kleisli map $C \rightarrow TA$ in \mathcal{C} . A parallel pair of terms $t \equiv t' : C \rightarrow TA$ is called an *equation*.

Definition 1.2 (Monadic Equational Systems). A *MES* $\mathbb{S} = (\underline{\mathcal{C}}, \mathbf{T}, \mathcal{A})$ consists of a strong monad \mathbf{T} on a symmetric monoidal closed category $\underline{\mathcal{C}}$ together with a set of equations \mathcal{A} .

Notation. For a strong monad \mathbf{T} on a symmetric monoidal closed category $\underline{\mathcal{C}}$ we implicitly assume that the respective underlying structures are denoted by $(T, \eta, \mu, \mathbf{st})$ and $(\mathcal{C}, I, \otimes, [-, =])$.

1.2. Model theory. Terms admit interpretations in algebras and these give a model-theoretic notion of equality.

Let (T, st) be a strong endofunctor on a symmetric monoidal closed category $\underline{\mathcal{C}}$. Every term $t : C \rightarrow TA$ admits an *interpretation*

$$\llbracket t \rrbracket_{(X,s)} : [A, X] \otimes C \rightarrow X$$

in a T -algebra $(X, s : TX \rightarrow X)$ given by the composite

$$[A, X] \otimes C \xrightarrow{[A, X] \otimes t} [A, X] \otimes TA \xrightarrow{\text{st}_{[A, X], A}} T([A, X] \otimes A) \xrightarrow{T(\epsilon_X^A)} TX \xrightarrow{s} X .$$

We thus obtain a *satisfaction relation* between algebras and equations: for all T -algebras (X, s) and equations $u \equiv v : C \rightarrow TA$,

$$(X, s) \models u \equiv v : C \rightarrow TA \quad \text{iff} \quad \llbracket u \rrbracket_{(X,s)} = \llbracket v \rrbracket_{(X,s)} : [A, X] \otimes C \rightarrow X .$$

Definition 1.3 (Algebras). An \mathbb{S} -algebra for a MES $\mathbb{S} = (\underline{\mathcal{C}}, \mathbf{T}, \mathcal{A})$ is an Eilenberg-Moore algebra (X, s) for the monad \mathbf{T} satisfying the equations in \mathcal{A} ; that is, such that $(X, s) \models u \equiv v : C \rightarrow TA$ for all $(u \equiv v : C \rightarrow TA) \in \mathcal{A}$.

The category $\mathbb{S}\text{-Alg}$ is the full subcategory of the category $\mathcal{C}^{\mathbf{T}}$ (of Eilenberg-Moore algebras for the monad \mathbf{T}) consisting of the \mathbb{S} -algebras. We thus have the following situation

$$\begin{array}{ccccc} \mathbb{S}\text{-Alg} & \hookrightarrow & \mathcal{C}^{\mathbf{T}} & \hookrightarrow & T\text{-Alg} \\ & \searrow & \downarrow U_{\mathbf{T}} & \swarrow & \\ & & \mathcal{C} & & \end{array}$$

$U_{\mathbb{S}} \quad U_T$

where $T\text{-Alg}$ denotes the category of algebras for the endofunctor T .

2. EQUATIONAL METALOGIC

We present a sound deductive system for reasoning about the equality of terms in MESs.

2.1. Equational Metalogic. The *Equational Metalogic* (*EML*) associated to a MES $\mathbb{S} = (\underline{\mathcal{C}}, \mathbf{T}, \mathcal{A})$ has judgements of the form

$$\mathcal{A} \vdash u \equiv v : C \rightarrow TA ,$$

where u and v are terms of arity A and coarity C , and consists of the following inference rules.

- Equality rules.

$$\begin{array}{c} \text{Ref} \frac{}{\mathcal{A} \vdash u \equiv u : C \rightarrow TA} \qquad \text{Sym} \frac{\mathcal{A} \vdash u \equiv v : C \rightarrow TA}{\mathcal{A} \vdash v \equiv u : C \rightarrow TA} \\ \\ \text{Trans} \frac{\mathcal{A} \vdash u \equiv v : C \rightarrow TA \quad \mathcal{A} \vdash v \equiv w : C \rightarrow TA}{\mathcal{A} \vdash u \equiv w : C \rightarrow TA} \end{array}$$

- Axioms.

$$\text{Axiom} \frac{(u \equiv v : C \rightarrow TA) \in \mathcal{A}}{\mathcal{A} \vdash u \equiv v : C \rightarrow TA}$$

- Congruence of substitution.

$$\text{Subst} \frac{\mathcal{A} \vdash u_1 \equiv v_1 : C \rightarrow TB \quad \mathcal{A} \vdash u_2 \equiv v_2 : B \rightarrow TA}{\mathcal{A} \vdash u_1\{u_2\} \equiv v_1\{v_2\} : C \rightarrow TA}$$

where $w_1\{w_2\}$ denotes the Kleisli composite $C \xrightarrow{w_1} TB \xrightarrow{T(w_2)} T(TA) \xrightarrow{\mu_A} TA$.

- Congruence of tensor extension.

$$\text{Ext} \frac{\mathcal{A} \vdash u \equiv v : C \rightarrow TA}{\mathcal{A} \vdash \langle V \rangle u \equiv \langle V \rangle v : V \otimes C \rightarrow T(V \otimes A)}$$

where $\langle V \rangle w$ denotes the composite $V \otimes C \xrightarrow{V \otimes w} V \otimes TA \xrightarrow{\text{st}_{V,A}} T(V \otimes A)$.

- Local character.

$$\text{Local} \frac{\mathcal{A} \vdash u \circ e_i \equiv v \circ e_i : C_i \rightarrow TA \quad (i \in I)}{\mathcal{A} \vdash u \equiv v : C \rightarrow TA} \quad (\{e_i : C_i \rightarrow C\}_{i \in I} \text{ jointly epi})$$

(Recall that a family of maps $\{e_i : C_i \rightarrow C\}_{i \in I}$ is said to be *jointly epi* if, for any $f, g : C \rightarrow X$ such that $\forall_{i \in I} f \circ e_i = g \circ e_i : C_i \rightarrow X$, it follows that $f = g$.)

Remark 2.1. In the presence of coproducts and under the rule Ref, the rules Subst and Local are inter-derivable with the rules

$$\text{Subst}_{\text{II}} \frac{\mathcal{A} \vdash u \equiv v : C \rightarrow T(\coprod_{i \in I} B_i) \quad \mathcal{A} \vdash u_i \equiv v_i : B_i \rightarrow TA \quad (i \in I)}{\mathcal{A} \vdash u\{[u_i]_{i \in I}\} \equiv v\{[v_i]_{i \in I}\} : C \rightarrow TA}$$

and

$$\text{Local}_1 \frac{\mathcal{A} \vdash u \circ e \equiv v \circ e : C' \rightarrow TA}{\mathcal{A} \vdash u \equiv v : C \rightarrow TA} \quad (e : C' \twoheadrightarrow C \text{ epi})$$

Indeed, consider the rule

$$\text{Local}_{\text{II}} \frac{\mathcal{A} \vdash u_i \equiv v_i : C_i \rightarrow TA \quad (i \in I)}{\mathcal{A} \vdash [u_i]_{i \in I} \equiv [v_i]_{i \in I} : \coprod_{i \in I} C_i \rightarrow TA}$$

and note that: (i) the rule Local is derivable from the rules Local₁ and Local_{II}, which are in turn instances of the rule Local; (ii) the rule Subst_{II} is derivable from the rules Subst and Local_{II}; (iii) the rule Subst is an instance of the rule Subst_{II}; and (iv) assuming the rule Ref, the rule Local_{II} is derivable from the rule Subst_{II}.

2.2. Soundness. The following result states the soundness of derivability in EML. We write $\mathbb{S}\text{-Alg} \models u \equiv v : C \rightarrow TA$ whenever $(X, s) \models u \equiv v : C \rightarrow TA$ for all \mathbb{S} -algebras (X, s) .

Theorem 2.2 (Soundness). *For a MES $\mathbb{S} = (\mathcal{C}, \mathbf{T}, \mathcal{A})$,*

$$\mathcal{A} \vdash u \equiv v : C \rightarrow TA \quad \text{implies} \quad \mathbb{S}\text{-Alg} \models u \equiv v : C \rightarrow TA .$$

Proof. See Appendix A. □

3. INTERNAL COMPLETENESS

In this section we build a mathematical basis for investigating completeness. Our main tools are an internal completeness result for MESs that admit free algebras together with an inductive method for constructing them.

3.1. Internal Completeness. Let $\mathbb{S} = (\mathcal{C}, \mathbf{T}, \mathcal{A})$ be a MES admitting free algebras; that is, such that the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-Alg} \rightarrow \mathcal{C}$ has a left adjoint. We denote the free \mathbb{S} -algebra on $X \in \mathcal{C}$ as $(T_{\mathbb{S}}X, \tau_X^{\mathbb{S}} : TT_{\mathbb{S}}X \rightarrow T_{\mathbb{S}}X)$, and the associated *free \mathbb{S} -algebra monad* as $\mathbf{T}_{\mathbb{S}} = (T_{\mathbb{S}}, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$. Then, the embedding $\mathbb{S}\text{-Alg} \hookrightarrow \mathcal{C}^{\mathbf{T}}$ induces a strong monad morphism $\mathbf{q}^{\mathbb{S}} : \mathbf{T} \rightarrow \mathbf{T}_{\mathbb{S}}$. This has components referred to as *quotient maps* that are characterised by being the unique morphisms $\mathbf{q}_X^{\mathbb{S}} : TX \rightarrow T_{\mathbb{S}}X$ for which the diagram

$$\begin{array}{ccc} TT X & \xrightarrow{T(\mathbf{q}_X^{\mathbb{S}})} & TT_{\mathbb{S}} X \\ \mu_X \downarrow & & \downarrow \tau_X^{\mathbb{S}} \\ TX & \xrightarrow{\mathbf{q}_X^{\mathbb{S}}} & T_{\mathbb{S}} X \\ \eta_X \uparrow & \nearrow \eta_X^{\mathbb{S}} & \\ X & & \end{array} \quad (3.1)$$

commutes. In this situation, we have a form of *strong completeness* stating that an equation is satisfied in all models if and only if it is satisfied in a freely generated one, if and only if it is identified by the quotient map.

Theorem 3.1 (Internal completeness). *For a MES $\mathbb{S} = (\mathcal{C}, \mathbf{T}, \mathcal{A})$ admitting free algebras, the following are equivalent.*

- (1) $\mathbb{S}\text{-Alg} \models u \equiv v : C \rightarrow TA$.
- (2) $(T_{\mathbb{S}}A, \tau_A^{\mathbb{S}}) \models u \equiv v : C \rightarrow TA$.
- (3) $\mathbf{q}_A^{\mathbb{S}} \circ u = \mathbf{q}_A^{\mathbb{S}} \circ v : C \rightarrow T_{\mathbb{S}}A$.

Proof. See Appendix B. □

3.2. Free Algebras. We now establish a general setting in which to apply the internal completeness theorem. Indeed, we give conditions under which MESs admit free algebras and provide an inductive construction of quotient maps (see Fiore and Hur (2009) for details).

Definition 3.2. Let \mathcal{C} be a symmetric monoidal closed category. An object A in \mathcal{C} is respectively said to be *compact* and *projective* if the endofunctor $[A, -]$ on \mathcal{C} respectively preserves colimits of ω -chains and epimorphisms.

Definition 3.3. A MES $\mathbb{S} = (\mathcal{C}, \mathbf{T}, \mathcal{A})$ is called *finitary* if the category \mathcal{C} is cocomplete, the endofunctor T on \mathcal{C} is ω -cocontinuous, and the arity A of each equation $u \equiv v : C \rightarrow TA$ in \mathcal{A} is compact. Such a MES is called *inductive* if furthermore the endofunctor T preserves epimorphisms and the arity A of each equation $u \equiv v : C \rightarrow TA$ in \mathcal{A} is projective.

For a finitary MES $\mathbb{S} = (\mathcal{C}, \mathbf{T}, \mathcal{A})$ we have the following situation:

$$\begin{array}{ccc} & \xleftarrow{K} & \mathcal{C}^{\mathbf{T}} \\ \mathbb{S}\text{-Alg} & \xrightleftharpoons[\quad J \quad]{\perp} & \mathcal{C}^{\mathbf{T}} \\ & & \downarrow U_{\mathbf{T}} \\ & & \mathcal{C} \end{array}$$

For each object $X \in \mathcal{C}$, since (TX, μ_X) is a free Eilenberg-Moore algebra on X , the free \mathbb{S} -algebra $(T_{\mathbb{S}}X, \tau_X^{\mathbb{S}})$ on X is given by the free \mathbb{S} -algebra $K(TX, \mu_X)$ over the Eilenberg-Moore algebra (TX, μ_X) . Satisfying the commutative diagram (3.1), the universal homomorphism $(TX, \mu_X) \rightarrow (T_{\mathbb{S}}X, \tau_X^{\mathbb{S}})$ induced by the adjunction $K \dashv J$ yields the quotient map $\mathfrak{q}_X^{\mathbb{S}} : TX \rightarrow T_{\mathbb{S}}X$.

In the case of inductive MESs, the quotient maps $\mathfrak{q}_X^{\mathbb{S}}$ are constructed as follows:

$$\begin{array}{ccccccc} \forall (u \equiv v : C \rightarrow TA) \in \mathcal{A} & T(TX) & \xrightarrow{T(q_0)} & T(TX)_1 & \xrightarrow{T(q_1)} & T(TX)_2 & \xrightarrow{T(q_2)} & T(TX)_3 & \cdots & T(T_{\mathbb{S}}X) \\ \vdots & \downarrow \mu_X & \searrow p_0 & \text{po} & \searrow p_1 & \text{po} & \searrow p_2 & & & \downarrow \tau_X^{\mathbb{S}} \\ [A, TX] \otimes C & \xrightarrow{[u]_{(TX, \mu_X)}} & TX & \xrightarrow[q_0]{\text{coeq}} & (TX)_1 & \xrightarrow{q_1} & (TX)_2 & \xrightarrow{q_2} & (TX)_3 & \cdots & T_{\mathbb{S}}X \\ \vdots & \xrightarrow{[v]_{(TX, \mu_X)}} & & & & & & & & & \text{colim} \end{array} \quad (3.2)$$

$\xrightarrow{\mathfrak{q}_X^{\mathbb{S}}}$

where q_0 is the universal map that jointly coequalizes every pair $[u]_{(TX, \mu_X)}$ and $[v]_{(TX, \mu_X)}$ with $(u \equiv v : C \rightarrow TA) \in \mathcal{A}$ and where, for all $n \geq 1$, the cospans

$$(TX)_n \xrightarrow{q_n} (TX)_{n+1} \xleftarrow{p_n} T(TX)_n$$

are pushouts of the spans

$$(TX)_n \xleftarrow{p_{n-1}} T(TX)_{n-1} \xrightarrow{T(q_{n-1})} T(TX)_n$$

for $(TX)_0 = TX$.

Moreover, when the strong monad \mathbf{T} arises from a left adjoint to a forgetful functor $F\text{-Alg} \rightarrow \mathcal{C}$, for F a strong endofunctor that preserves colimits of ω -chains and epimorphisms, the construction of the quotient maps $q_X^{\mathbb{S}}$ simplifies as follows:

$$\begin{array}{ccccccc}
 \forall (u \equiv v : C \rightarrow TA) \in \mathcal{A} & & F(TX) & \xrightarrow{F(q_0)} & F(TX)_1 & \xrightarrow{F(q_1)} & F(TX)_2 & \xrightarrow{F(q_2)} & F(TX)_3 & \cdots & F(T_{\mathbb{S}}X) \\
 \vdots & & \downarrow \hat{\mu}_X & \searrow p_0 & & \searrow p_1 & & \searrow p_2 & & & \downarrow \hat{\tau}_X^{\mathbb{S}} \\
 [A, TX] \otimes C & \xrightarrow{\llbracket u \rrbracket_{(TX, \mu_X)}} & TX & \xrightarrow[q_0]{\text{coeq}} & (TX)_1 & \xrightarrow{q_1} & (TX)_2 & \xrightarrow{q_2} & (TX)_3 & \cdots & T_{\mathbb{S}}X \\
 \vdots & & \downarrow \llbracket v \rrbracket_{(TX, \mu_X)} & & & & & & & & \downarrow \text{colim} \\
 & & & & & & & & & & q_X^{\mathbb{S}}
 \end{array} \tag{3.3}$$

where $(TX, \hat{\mu}_X)$ and $(T_{\mathbb{S}}X, \hat{\tau}_X^{\mathbb{S}})$ are the F -algebras respectively corresponding to the Eilenberg-Moore algebras (TX, μ_X) and $(T_{\mathbb{S}}X, \tau_X^{\mathbb{S}})$ for the monad \mathbf{T} . (Explicit calculations of this construction feature in Sections 4.6 and 5.7.)

PART II. METHODOLOGY

In view of the mathematical development of Part I, we advocate the following methodology for synthesising mono-sorted equational logics.

- (1) *Select a symmetric monoidal closed category \mathcal{C} as universe of discourse and consider within it a syntactic notion of signature such that every signature Σ gives rise to a strong monad \mathbf{T}_{Σ} on \mathcal{C} .*

The universe of discourse should be carefully chosen to consist of mathematical objects with enough internal structure to allow for the algebraic realisation of the syntactic constructs that one is modelling.

We do not insist on an a priori prescription for the definition of signature, but rather consider it as being domain specific. Of course, standard notions of signature (*e.g.* as in enriched algebraic theories—see Kelly and Power (1993), and Robinson (2002)) may be considered. However, one may need to go beyond them—see Fiore (2008) and Fiore and Hur (2010).

- (2) *Select a class of coarity-arity pairs (C, A) of objects of \mathcal{C} and give a syntactic description of Kleisli maps $C \rightarrow T_{\Sigma}A$. This yields a syntactic notion of equational presentation with an associated model theory arising from that of MESs.*

We are ultimately interested in constructing free algebras for equational presentations. In the context of finitary algebraic theories, it is thus appropriate to consider a co-complete universe of discourse together with signatures for which the associated monad preserves colimits of ω -chains and epimorphisms, and arities that are compact and projective; so that the induced MESs are inductive.

- (3) *Synthesise a deductive system for equational reasoning on syntactic terms with rules arising as syntactic counterparts of the EML rules associated to the MES.*

The analysis of the rule **Subst** will typically involve the consideration of a syntactic substitution operation corresponding to Kleisli composition.

- (4) *Analyse the inductive construction of free algebras and obtain an intermediate deductive system characterising the equivalence induced by the quotient maps. Embed the intermediate deductive system within the synthesised equational logic and conclude the completeness of the latter as a consequence of the internal completeness result.*

In practise, we have found that the intermediate deductive system is not only easily embeddable in the synthesised equational logic but that it moreover allows one to distil a rewriting-style deduction system that provides a sound and complete computational treatment of derivability.

The resulting equational logics are thus synthesised from algebraic metatheories by means of first principles. Two sample applications of this methodology follow.

4. SYNTHETIC EQUATIONAL LOGIC

4.1. MESs for algebraic theories. Recall that an *algebraic theory* $\mathbb{T} = (\Sigma, E)$ is given by a signature Σ , consisting of a set of operators O and an arity function $|-| : O_\Sigma \rightarrow \mathbb{N}$, together with a set of equations E . Algebraic theories may be encoded as MESs as follows.

The signature Σ induces the endofunctor $F_\Sigma(X) = \coprod_{o \in O} X^{|o|}$ on \mathbf{Set} , for which the category of Σ -algebras, $\Sigma\text{-Alg}$, and the category of F_Σ -algebras, $F_\Sigma\text{-Alg}$, are isomorphic. The forgetful functor $F_\Sigma\text{-Alg} \rightarrow \mathbf{Set}$ is monadic and the induced *term monad* $\mathbf{T}_\Sigma = (T_\Sigma, \eta^\Sigma, \mu^\Sigma)$ is given syntactically. For a set of variables V , the set $T_\Sigma(V)$ consists of terms built up from the variables in V and the operators in O .

The endofunctor F_Σ has a canonical strength $\mathbf{st} : U \times F_\Sigma(V) \rightarrow F_\Sigma(U \times V)$ mapping a pair $(u, \iota_o(v_1, \dots, v_{|o|}))$ to $\iota_o((u, v_1), \dots, (u, v_{|o|}))$, where we use the notation ι for coproduct injections. The induced strength on the monad \mathbf{T}_Σ , $\widehat{\mathbf{st}} : U \times T_\Sigma(V) \rightarrow T_\Sigma(U \times V)$, maps a pair (u, t) to the term $t\{v \mapsto (u, v)\}_{v \in V}$ obtained by simultaneously substituting (u, v) for each variable $v \in V$ in the term t .

By definition, each equation $(V \vdash l \equiv r)$ in E is given by a pair of terms $l, r \in T_\Sigma(V)$, or equivalently, by a parallel pair of Kleisli maps $\underline{l}, \underline{r} : 1 \rightarrow T_\Sigma(V)$. Thus, one can encode the algebraic theory \mathbb{T} as the MES $\underline{\mathbb{T}} = (\mathbf{Set}, \mathbf{T}_\Sigma, \underline{E})$ with the set of equations \underline{E} given by $\{\underline{l} \equiv \underline{r} : 1 \rightarrow T_\Sigma V \mid (V \vdash l \equiv r) \in E\}$. The MES $\underline{\mathbb{T}}$ is inductive.

4.2. Model theory. A $\underline{\mathbb{T}}$ -algebra is an Eilenberg-Moore algebra $(X, s : T_\Sigma X \rightarrow X)$ for \mathbf{T}_Σ such that the diagram

$$X^V \times 1 \begin{array}{c} \xrightarrow{X^V \times \underline{t}_1} \\ \xrightarrow{X^V \times \underline{t}_2} \end{array} X^V \times T_\Sigma V \xrightarrow{\widehat{\mathbf{st}}_{X^V, V}} T_\Sigma(X^V \times V) \xrightarrow{T_\Sigma(\epsilon_X^V)} T_\Sigma X \xrightarrow{s} X$$

commutes for every equation $(V \vdash t_1 \equiv t_2)$ in E ; that is, such that

$$1 \begin{array}{c} \xrightarrow{\underline{t}_1} \\ \xrightarrow{\underline{t}_2} \end{array} T_\Sigma V \xrightarrow{T_\Sigma(v)} T_\Sigma X \xrightarrow{s} X \quad (4.1)$$

commutes for all functions $v : V \rightarrow X$.

Write $(X, \llbracket - \rrbracket)$ for the Eilenberg-Moore algebra of the monad \mathbf{T}_Σ corresponding to the Σ -algebra $(X, \{\llbracket o \rrbracket\}_{o \in \Sigma})$ via the isomorphism $\Sigma\text{-Alg} \cong \mathcal{C}^{\mathbf{T}_\Sigma}$. We have that the Eilenberg-Moore algebra $(X, \llbracket - \rrbracket)$ satisfies (4.1) if and only if the Σ -algebra $(X, \{\llbracket o \rrbracket\}_{o \in \Sigma})$ satisfies the

equation ($V \vdash t_1 \equiv t_2$). It follows thus that $\underline{\mathbb{T}}\text{-Alg}$ is isomorphic to the category $\mathbb{T}\text{-Alg}$ of algebras for the algebraic theory \mathbb{T} .

4.3. EML for algebraic theories. The EML associated to the MES of an algebraic theory $\mathbb{T} = (\Sigma, E)$ has judgements of the form

$$\underline{E} \vdash f \equiv g : U \rightarrow T_\Sigma V$$

with inference rules Ref, Sym, Trans, Axiom, Subst $_{\Pi}$, Ext, and Local $_1$ (see Section 2). The rules Ext and Local $_1$ are however redundant. Indeed, the subsystem EML $_1$ with inference rules Ref, Sym, Trans, Axiom, and Subst $_{\Pi}$ restricted to judgements of the form

$$\underline{E} \vdash u \equiv v : 1 \rightarrow T_\Sigma V$$

is such that

$$\begin{aligned} & \underline{E} \vdash f \equiv g : U \rightarrow T_\Sigma V \text{ is derivable in EML} \\ \text{iff} & \\ & \underline{E} \vdash f\{\underline{i}\} \equiv g\{\underline{i}\} : 1 \rightarrow T_\Sigma V \text{ is derivable in EML}_1 \text{ for all } i \in U. \end{aligned}$$

4.4. Synthetic Equational Logic. A Synthetic Equational Logic (SEL) for algebraic theories $\mathbb{T} = (\Sigma, E)$ directly arises as the syntactic counterpart of EML $_1$. SEL has judgements

$$V \vdash_E s \equiv t \quad (s, t \in T_\Sigma V)$$

and consists of the following rules:

$$\begin{aligned} \text{Ref} & \frac{}{V \vdash_E t \equiv t} & \text{Sym} & \frac{V \vdash_E t \equiv t'}{V \vdash_E t' \equiv t} & \text{Trans} & \frac{V \vdash_E t \equiv t' \quad V \vdash_E t' \equiv t''}{V \vdash_E t \equiv t''} \\ & & \text{Axiom} & \frac{(V \vdash l \equiv r) \in E}{V \vdash_E l \equiv r} & & \\ & & \text{Subst} & \frac{U \vdash_E t \equiv t' \quad V \vdash_E s_u \equiv s'_u \ (u \in U)}{V \vdash_E t\{u \mapsto s_u\}_{u \in U} \equiv t'\{u \mapsto s'_u\}_{u \in U}} & & \end{aligned}$$

In the rule Subst, the term $t\{u \mapsto s_u\}_{u \in U}$ is obtained by simultaneously substituting the terms s_u for the variables $u \in U$ in the term t .

4.5. Soundness. Note that SEL subsumes the usual presentation of Equational Logic, where the substitution rule is restricted to families $s_u = s'_u$ ($u \in U$) and a congruence rule for operators is added. Furthermore, since $V \vdash_E s \equiv t$ is derivable in SEL iff $\underline{E} \vdash \underline{s} \equiv \underline{t} : 1 \rightarrow T_\Sigma V$ is derivable in EML $_1$ iff $\underline{E} \vdash \underline{s} \equiv \underline{t} : 1 \rightarrow T_\Sigma V$ is derivable in EML, the well-known soundness of SEL follows from the soundness of EML.

4.6. Completeness. We proceed to show how the internal completeness theorem and the construction of free algebras for inductive MESs (see Section 3) lead to equational derivability and bidirectional rewriting completeness results.

Consider the construction (3.3) for the MES $\underline{\mathbb{T}}$. The map $q_0 : T_\Sigma X \twoheadrightarrow (T_\Sigma X)_1$ is the universal map in **Set** that coequalizes every pair $[[l], [r]] : (T_\Sigma X)^V \rightarrow T_\Sigma X$ for all $(V \vdash l \equiv r) \in E$, where $[[t]]$ maps $s \in (T_\Sigma X)^V$ to $t\{v \mapsto s_v\}_{v \in V}$. It follows that the set $(T_\Sigma X)_1$ is given by the quotient $T_\Sigma X / \approx_1$ of $T_\Sigma X$ under the equivalence relation \approx_1 generated by the rule:

$$\frac{(V \vdash l \equiv r) \in E}{l\{v \mapsto s_v\}_{v \in V} \approx_1 r\{v \mapsto s_v\}_{v \in V}} \quad (s \in (T_\Sigma X)^V)$$

The map q_0 sends a term $t \in T_\Sigma X$ to its equivalence class $[t]_{\approx_1} \in T_\Sigma X / \approx_1$, and the map p_0 sends $\iota_{\mathbf{o}}(t_1, \dots, t_{|\mathbf{o}|}) \in F_\Sigma(T_\Sigma X)$ to $[\mathbf{o}(t_1, \dots, t_{|\mathbf{o}|})]_{\approx_1} \in T_\Sigma X / \approx_1$.

Recall that a pushout

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ & \searrow f & \downarrow f' \\ & & C \xrightarrow{e'} C/\approx \end{array} \quad \text{po}$$

of a surjective map $e : A \twoheadrightarrow B$ and a map $f : A \rightarrow C$ in **Set** can be constructed as the cospan $e' : C \twoheadrightarrow C/\approx \leftarrow B : f'$, where C/\approx is the quotient of C under the equivalence relation \approx generated by setting $f(a) \approx f(a')$ for all $a, a' \in A$ such that $e(a) = e(a')$ in B , and where the surjective map $e' : C \twoheadrightarrow C/\approx$ sends an element c to its equivalence class $[c]_{\approx}$ and the map $f' : B \rightarrow C/\approx$ sends an element b to $e'(f(a))$ for $a \in A$ such that $e(a) = b$.

Using this construction, an inductive analysis of the maps q_n for $n \geq 1$ shows that the sets $(T_\Sigma X)_n$ for $n \geq 2$ are given as the quotients $T_\Sigma X / \approx_n$ of $T_\Sigma X$ under the equivalence relations \approx_n inductively generated by the following rules:

$$\frac{s \approx_{n-1} s'}{s \approx_n s'} \quad \frac{s_1 \approx_{n-1} s'_1, \dots, s_{|\mathbf{o}|} \approx_{n-1} s'_{|\mathbf{o}|}}{\mathbf{o}(s_1, \dots, s_{|\mathbf{o}|}) \approx_n \mathbf{o}(s'_1, \dots, s'_{|\mathbf{o}|})} \quad (\mathbf{o} \in \Sigma)$$

The maps q_n for $n \geq 1$ send $[t]_{\approx_n} \in T_\Sigma X / \approx_n$ to $[t]_{\approx_{n+1}} \in T_\Sigma X / \approx_{n+1}$, and the maps p_n for $n \geq 1$ send $\iota_{\mathbf{o}}([t_1]_{\approx_n}, \dots, [t_{|\mathbf{o}|}]_{\approx_n}) \in F_\Sigma(T_\Sigma X / \approx_n)$ to $[\mathbf{o}(t_1, \dots, t_{|\mathbf{o}|})]_{\approx_{n+1}} \in T_\Sigma X / \approx_{n+1}$.

By taking the colimit of the chain of quotients $\{q_n : T_\Sigma X / \approx_n \twoheadrightarrow T_\Sigma X / \approx_{n+1}\}_{n \geq 0}$, the set $T_{\mathbb{S}}X$ is given by the quotient $T_\Sigma X / \approx_E$ of $T_\Sigma X$ under the relation \approx_E generated by the following rules:

$$\begin{array}{l} \text{Ref } \frac{}{s \approx_E s} \quad \text{Sym } \frac{s \approx_E s'}{s' \approx_E s} \quad \text{Trans } \frac{s \approx_E s' \quad s' \approx_E s''}{s \approx_E s''} \\ \text{Inst } \frac{(V \vdash l \equiv r) \in E}{l\{v \mapsto s_v\}_{v \in V} \approx_E r\{v \mapsto s_v\}_{v \in V}} \quad (s \in (T_\Sigma X)^V) \\ \text{Cong } \frac{s_1 \approx_E s'_1, \dots, s_{|\mathbf{o}|} \approx_E s'_{|\mathbf{o}|}}{\mathbf{o}(s_1, \dots, s_{|\mathbf{o}|}) \approx_E \mathbf{o}(s'_1, \dots, s'_{|\mathbf{o}|})} \quad (\mathbf{o} \in \Sigma) \end{array} \quad (4.2)$$

The quotient map $q_X^{\mathbb{S}} : T_\Sigma X \twoheadrightarrow T_{\mathbb{S}}X$ sends a term t to its equivalence class $[t]_{\approx_E}$.

The rules **Inst** and **Cong** for the relation \approx_E can be merged into a single rule to yield a rewriting-style deduction system. Indeed, by an induction on the depth of proof trees, one shows that the relation \approx_E on $T_\Sigma X$ coincides with the equivalence relation \approx_E^R on $T_\Sigma X$ generated by the rewriting-style rule

$$\mathbf{C}[l\{v \mapsto s_v\}_{v \in V}] \approx_E^R \mathbf{C}[r\{v \mapsto s_v\}_{v \in V}]$$

for $(V \vdash l \equiv r) \in E$, $s \in (T_\Sigma X)^V$, and $\mathbf{C}[-]$ a context with one hole and possibly with variables from X .

From the internal completeness of the MES $\underline{\mathbb{T}}$, we have the soundness and completeness of equational reasoning by bidirectional rewriting:

$$\begin{aligned} \mathbb{T}\text{-Alg} &\models (V \vdash s \equiv t) \\ \iff \underline{\mathbb{T}}\text{-Alg} &\models (\underline{s} \equiv \underline{t} : 1 \rightarrow T_\Sigma V) \\ \iff \mathbf{q}_V^{\mathbb{S}} \circ \underline{s} &= \mathbf{q}_V^{\mathbb{S}} \circ \underline{t} : 1 \rightarrow T_{\mathbb{S}} V \\ \iff [s]_{\approx_E} &= [t]_{\approx_E} \text{ in } T_\Sigma V / \approx_E \\ \iff s \approx_E t &\text{ in } T_\Sigma V \\ \iff s \approx_E^R t &\text{ in } T_\Sigma V \end{aligned}$$

Finally, also SEL is complete; as a proof of $s \approx_E t$ for $s, t \in T_\Sigma V$ constructed by the rules in (4.2) can be turned into a proof of $V \vdash_E s \equiv t$ in SEL.

5. SYNTHETIC NOMINAL EQUATIONAL LOGIC

This section provides a novel application of our theory and methodology for synthesising equational logics geared to the development of a deductive system for reasoning about algebraic structure with name-binding operators.

We consider a class of MESs, referred to as *Nominal Equational Systems (NESs)*, based on the category **Nom** of nominal sets (Gabbay and Pitts, 2001, Section 6) (or equivalently the Schanuel topos (Mac Lane and Moerdijk, 1992, Section III.9)). These we subsequently present in syntactic form to yield *Nominal Equational Presentations (NEPs)*. The model theory of NEPs is of course derived from that of NESs.

An equational logic, called *Synthetic Nominal Equational Logic (SNEL)*, for NEPs is derived from the EML associated to NESs. This is guaranteed to be sound by construction. Completeness is derived from the internal completeness theorem by an analysis of the inductive construction of free algebras in terms of equational derivability. This approach yields two completeness results: the rewriting completeness of an induced notion of *Synthetic Nominal Rewriting (SNR)* and the derivability completeness of SNEL.

A brief discussion of related work is included.

5.1. Nominal sets. For a fixed countably infinite set A of atoms, the group $\mathfrak{S}_0(A)$ of finite permutations of atoms consists of the bijections on A that fix all but finitely many elements of A . A $\mathfrak{S}_0(A)$ -action $X = (|X|, \cdot)$ consists of a set $|X|$ equipped with a function $(-) \cdot (=) : \mathfrak{S}_0(A) \times |X| \rightarrow |X|$ satisfying $\text{id}_A \cdot x = x$ and $\pi' \cdot (\pi \cdot x) = (\pi' \pi) \cdot x$ for all $x \in |X|$ and $\pi, \pi' \in \mathfrak{S}_0(A)$. $\mathfrak{S}_0(A)$ -actions form a category with morphisms $X \rightarrow Y$ given by *equivariant functions*; that is, functions $f : |X| \rightarrow |Y|$ such that $f(\pi \cdot x) = \pi \cdot (fx)$ for all $\pi \in \mathfrak{S}_0(A)$ and $x \in |X|$.

For a $\mathfrak{S}_0(\mathbb{A})$ -action X , a finite subset S of \mathbb{A} is said to *support* $x \in X$ if for all atoms $a, a' \notin S$, we have that $(a \ a') \cdot x = x$, where the *transposition* $(a \ a')$ is the bijection that swaps a and a' , and fixes all other atoms. A *nominal set* is a $\mathfrak{S}_0(\mathbb{A})$ -action in which every element has finite support. As an example, the set of atoms \mathbb{A} becomes the *nominal set of atoms* \mathbb{A} when equipped with the evaluation action $\pi \cdot a = \pi(a)$. The category **Nom** is the full subcategory of the category of $\mathfrak{S}_0(\mathbb{A})$ -actions consisting of nominal sets.

The supports of an element of a nominal set are closed under intersection, and we write $\text{supp}_X(x)$, or simply $\text{supp}(x)$, for the intersection of the supports of x in the nominal set X . For instance, we have that $\text{supp}_{\mathbb{A}}(a) = \{a\}$. For elements x and y of two, possibly distinct, nominal sets X and Y , we write $x \# y$ whenever $\text{supp}_X(x)$ and $\text{supp}_Y(y)$ are disjoint. Thus, for $a \in \mathbb{A}$ and $x \in X$, $a \# x$ stands for $a \notin \text{supp}_X(x)$; that is, a is *fresh* for x .

The category **Nom** is complete and cocomplete. In particular, for a possibly infinite family of nominal sets $\{X_i\}_{i \in I}$, the coproduct $\coprod_{i \in I} X_i$ is given by $|\coprod_{i \in I} X_i| = \coprod_{i \in I} |X_i|$ with action $\pi \cdot \iota_i(x) = \iota_i(\pi \cdot x)$, where we use the notation ι for coproduct injections; the product $\prod_{i \in I} X_i$, for a finite set I , is given by $|\prod_{i \in I} X_i| = \prod_{i \in I} |X_i|$ with action $\pi \cdot \{x_i\}_{i \in I} = \{\pi \cdot x_i\}_{i \in I}$. As usual, we write X^n for the n -fold product $X \times \cdots \times X$.

Further, **Nom** carries a symmetric monoidal closed structure $(1, \#, [-, =])$. The *unit* 1 is the terminal object in **Nom** (i.e., the singleton set consisting of the empty tuple equipped with the unique action). The *separating tensor* $X \# Y$ is the nominal subset of $X \times Y$ with underlying set given by $\{(x, y) \in |X| \times |Y| \mid x \# y\}$. We write $X^{\#n}$ for the n -fold tensor product $X \# \cdots \# X$. For instance, $\mathbb{A}^{\#n}$ consists of n -tuples of distinct atoms equipped with the pointwise action $\pi \cdot (a_1, \dots, a_n) = (\pi \cdot a_1, \dots, \pi \cdot a_n)$. Note that $X^{\#0}$ is 1 for any nominal set X . Henceforth we write \mathbf{a}^n , or simply \mathbf{a} when n is clear from the context, as a shorthand for a tuple a_1, \dots, a_n of distinct atoms, and further write $\{\mathbf{a}^n\}$ for the set $\{a_1, \dots, a_n\}$. For pairs $\mathbf{a}, \mathbf{b} \in \mathbb{A}^{\#n}$ we define the *multi-transposition* $(\mathbf{a} \ \mathbf{b})$ to be a *fixed* bijection on \mathbb{A} such that $(\mathbf{a} \ \mathbf{b})(a_i) = b_i$ for all $1 \leq i \leq n$, and $(\mathbf{a} \ \mathbf{b})(c) = c$ for all $c \notin \{\mathbf{a}\} \cup \{\mathbf{b}\}$.

The separating tensor $\#$ is closed and the associated internal-hom functor is denoted $[-, =]$. In particular, the internal homs $[\mathbb{A}^{\#n}, X]$, for $n \in \mathbb{N}$ and $X \in \mathbf{Nom}$, give rise to a notion of multi-atom abstraction. Indeed, the nominal set $[\mathbb{A}^{\#n}, X]$ has underlying set given by the quotient $|\mathbb{A}^{\#n} \times X| / \approx_\alpha$ under the α -*equivalence relation* \approx_α defined as follows: $(\mathbf{a}, x) \approx_\alpha (\mathbf{b}, y)$ if and only if there exists a *fresh* $\mathbf{c} \in \mathbb{A}^{\#n}$ (i.e., a tuple $\mathbf{c} \in \mathbb{A}^{\#n}$ satisfying the condition $\mathbf{c} \# \mathbf{a}, x, \mathbf{b}, y$) such that $(\mathbf{a} \ \mathbf{c}) \cdot x = (\mathbf{b} \ \mathbf{c}) \cdot y$. We write $\langle \mathbf{a} \rangle x$ for the equivalence class $[(\mathbf{a}, x)]_{\approx_\alpha}$. The nominal set $[\mathbb{A}^{\#n}, X]$ has action $\pi \cdot \langle \mathbf{a} \rangle x = \langle \pi \cdot \mathbf{a} \rangle \pi \cdot x$. Note that $\text{supp}(\langle \mathbf{a} \rangle x)$ is $\text{supp}(x) \setminus \{\mathbf{a}\}$.

5.2. Nominal Equational Systems. We specify a class of MESs on **Nom**, called *Nominal Equational Systems (NESs)*.

Following (Clouston and Pitts, 2007) we define a *nominal signature* Σ to be a family of nominal sets $\{\Sigma(n)\}_{n \in \mathbb{N}}$, each of which consists of the operators of arity n .

Example 5.1. The nominal signature Σ_λ for the untyped λ -calculus is given by the nominal sets of operators

$$\Sigma_\lambda(0) = \{ \mathbf{V}_a \mid a \in \mathbb{A} \}, \quad \Sigma_\lambda(1) = \{ \mathbf{L}_a \mid a \in \mathbb{A} \}, \quad \Sigma_\lambda(2) = \{ \mathbf{A} \}$$

with action

$$\pi \cdot \mathbf{V}_a = \mathbf{V}_{\pi(a)}, \quad \pi \cdot \mathbf{L}_a = \mathbf{L}_{\pi(a)}, \quad \pi \cdot \mathbf{A} = \mathbf{A} .$$

To each nominal signature Σ , we associate the strong endofunctor $(F_\Sigma, \text{st}^\Sigma)$ on $\mathbf{Nom} = (\mathbf{Nom}, 1, \#, [-, =])$ as follows:

$$\begin{aligned} F_\Sigma(X) &= \coprod_{n \in \mathbb{N}} \Sigma(n) \times X^n, \\ \text{st}_{X,Y}^\Sigma &: \quad F_\Sigma(X) \# Y \quad \rightarrow \quad F_\Sigma(X \# Y) \\ &\quad (\iota_n(\mathbf{o}, x_1, \dots, x_n), y) \quad \mapsto \quad \iota_n(\mathbf{o}, (x_1, y), \dots, (x_n, y)) \end{aligned}$$

for $X, Y \in \mathbf{Nom}$ and $n \in \mathbb{N}$, $\mathbf{o} \in \Sigma(n)$, $x_1, \dots, x_n \in X$, $y \in Y$. Since \mathbf{Nom} is cocomplete and the functor F_Σ is ω -cocontinuous, free F_Σ -algebras exist. The carrier of the free F_Σ -algebra $T_\Sigma X$ on X has the following inductive syntactic description:

$$\begin{aligned} t \in T_\Sigma X & ::= x && (x \in X) \\ & | \quad \mathbf{o}(t_1, \dots, t_n) && (\mathbf{o} \in \Sigma(n), t_1, \dots, t_n \in T_\Sigma X) \end{aligned}$$

with action given by $\pi \cdot x = \pi \cdot_X x$ and $\pi \cdot \mathbf{o}(t_1, \dots, t_n) = (\pi \cdot \mathbf{o})(\pi \cdot t_1, \dots, \pi \cdot t_n)$. The associated *term monad* $\mathbf{T}_\Sigma = (T_\Sigma, \eta^\Sigma, \mu^\Sigma)$ is strong, with strength $\widehat{\text{st}}^\Sigma$ given as follows:

$$\begin{aligned} \widehat{\text{st}}_{X,Y}^\Sigma &: \quad T_\Sigma(X) \# Y \quad \rightarrow \quad T_\Sigma(X \# Y) \\ &: \quad (t, y) \quad \mapsto \quad t\{x \mapsto (x, y)\}_{x \in X} \end{aligned}$$

where $t\{x \mapsto (x, y)\}_{x \in X}$ denotes the term obtained by simultaneously substituting (x, y) for each x in the term t .

A NES is a MES $(\mathbf{Nom}, \mathbf{T}_\Sigma, \mathcal{A})$ for Σ a nominal signature and \mathcal{A} a set of equations with arities $\coprod_{i=1}^\ell \mathbb{A}^{\#n_i}$ for $\ell, n_1, \dots, n_\ell \in \mathbb{N}$ and coarities $\mathbb{A}^{\#n}$ for $n \in \mathbb{N}$.

5.3. Nominal Equational Presentations. We introduce *Nominal Equational Presentations (NEPs)* as syntactic counterparts of NESs.

We define a *variable context* as a finite set of variables $|V|$ together with a function $V : |V| \rightarrow \mathbb{N}$ assigning a *valence* to each variable. We write $x_1 : n_1, \dots, x_\ell : n_\ell$ for the variable context with variables x_1, \dots, x_ℓ respectively of valence n_1, \dots, n_ℓ . Variable contexts are syntax for arities, and every such V determines the arity

$$\underline{V} = \coprod_{x \in |V|} \mathbb{A}^{\#V(x)}.$$

We write $x(\mathbf{a})$ for the element $\iota_x(\mathbf{a})$ of \underline{V} ; when convenient, we further abbreviate $x()$ as x .

For $n \in \mathbb{N}$ and a variable context V , the following bijection

$$\begin{aligned} \mathbf{Nom}(\mathbb{A}^{\#n}, T_\Sigma \underline{V}) & \\ & \cong \mathbf{Nom}(1, [\mathbb{A}^{\#n}, T_\Sigma \underline{V}]) \\ & \cong \{ \tau \in [\mathbb{A}^{\#n}, T_\Sigma \underline{V}] \mid \text{supp}(\tau) = \emptyset \} \\ & = \{ \langle \mathbf{a} \rangle t \in [\mathbb{A}^{\#n}, T_\Sigma \underline{V}] \mid \text{supp}(t) \subseteq \{ \mathbf{a} \} \} \end{aligned} \tag{5.1}$$

shows that a Kleisli map $\mathbb{A}^{\#n} \rightarrow T_\Sigma \underline{V}$ is determined by an α -equivalence class $\langle \mathbf{a} \rangle t$ for $\mathbf{a} \in \mathbb{A}^{\#n}$ and $t \in (T_\Sigma \underline{V})_{\mathbf{a}} = \{ t \in T_\Sigma \underline{V} \mid \text{supp}(t) \subseteq \{ \mathbf{a} \} \}$. The set $(T_\Sigma \underline{V})_{\mathbf{a}}$ has the following inductive syntactic description:

$$\begin{aligned} t \in (T_\Sigma \underline{V})_{\mathbf{a}} & ::= x(\mathbf{b}) && (x(\mathbf{b}) \in \underline{V} \text{ such that } \{ \mathbf{b} \} \subseteq \{ \mathbf{a} \}) \\ & | \quad \mathbf{o}(t_1, \dots, t_n) && (\mathbf{o} \in \Sigma(n) \text{ such that } \text{supp}(\mathbf{o}) \subseteq \{ \mathbf{a} \}, \\ & && \text{and } t_1, \dots, t_n \in (T_\Sigma \underline{V})_{\mathbf{a}}). \end{aligned}$$

Directly motivated by this analysis, we define the notion of *Nominal Equational Presentation (NEP)* as follows. A *nominal context* $[\mathbf{a}]V$ consists of an atom context $\mathbf{a} \in \mathbb{A}^{\#n}$, for $n \in \mathbb{N}$, and a variable context V . A *nominal term* t , for a nominal signature Σ , in a nominal context $[\mathbf{a}]V$, denoted $[\mathbf{a}]V \vdash t$, is given by a term $t \in (T_{\Sigma}V)_{\mathbf{a}}$; a *nominal equation* $[\mathbf{a}]V \vdash t \equiv t'$ is given by a pair of nominal terms t and t' in the same nominal context $[\mathbf{a}]V$. A NEP $\mathbb{T} = (\Sigma, E)$ consists of a nominal signature Σ and a set of nominal equations E .

Example 5.2 (continued from Example 5.1, *cf.* (Gabbay and Mathijssen, 2007) and (Clouston and Pitts, 2007)). The NEP $\mathbb{T}_{\lambda} = (\Sigma_{\lambda}, E_{\lambda})$ for $\alpha\beta\eta$ -equivalence of untyped λ -terms has the following equations:

$$\begin{aligned}
(\alpha) \quad & [a, b] x : 1 \vdash \mathbf{L}_a.x(a) \equiv \mathbf{L}_b.x(b) \\
(\beta_{\kappa}) \quad & [a] x : 0, y : 1 \vdash \mathbf{A}(\mathbf{L}_a.x, y(a)) \equiv x \\
(\beta_{\mathbf{V}}) \quad & [a] x : 1 \vdash \mathbf{A}(\mathbf{L}_a.\mathbf{V}_a, x(a)) \equiv x(a) \\
(\beta_{\mathbf{L}}) \quad & [a, b] x : 2, y : 1 \vdash \mathbf{A}(\mathbf{L}_a.\mathbf{L}_b.x(a, b), y(a)) \equiv \mathbf{L}_b.\mathbf{A}(\mathbf{L}_a.x(a, b), y(a)) \\
(\beta_{\mathbf{A}}) \quad & [a] x : 1, y : 1, z : 1 \vdash \mathbf{A}(\mathbf{L}_a.\mathbf{A}(x(a), y(a)), z(a)) \\
& \quad \quad \quad \equiv \mathbf{A}(\mathbf{A}(\mathbf{L}_a.x(a), z(a)), \mathbf{A}(\mathbf{L}_a.y(a), z(a))) \\
(\beta_{\varepsilon}) \quad & [a, b] x : 1 \vdash \mathbf{A}(\mathbf{L}_a.x(a), \mathbf{V}_b) \equiv x(b) \\
(\eta) \quad & [a] x : 0 \vdash \mathbf{L}_a.\mathbf{A}(x, \mathbf{V}_a) \equiv x
\end{aligned}$$

where we write $\mathbf{L}_a.t$ for $\mathbf{L}_a(t)$.

By construction, thus, NEPs represent NESs. Indeed, a NEP $\mathbb{T} = (\Sigma, E)$ determines the NES $\underline{\mathbb{T}} = (\underline{\mathbf{Nom}}, \underline{\mathbf{T}}_{\Sigma}, \underline{E})$ with the set of equations \underline{E} given by

$$\{ \underline{[\mathbf{a}]V \vdash l} \equiv \underline{[\mathbf{a}]V \vdash r} : \mathbb{A}^{\#n} \rightarrow T_{\Sigma}V \}_{([\mathbf{a}]V \vdash l \equiv r) \in E}$$

where $\underline{[\mathbf{a}]V \vdash t}$ is the Kleisli map corresponding to $\langle \mathbf{a} \rangle t$ via the bijection (5.1).

5.4. Model theory. The model theory of a NEP $\mathbb{T} = (\Sigma, E)$ is derived from that of the NES $\underline{\mathbb{T}}$. This we now spell out in elementary terms.

A \mathbb{T} -algebra is an F_{Σ} -algebra $(M, \mathbf{e} : F_{\Sigma}M \rightarrow M)$ such that for all nominal equations $([\mathbf{a}]V \vdash l \equiv r) \in E$,

$$\llbracket [\mathbf{a}]V \vdash l \rrbracket_{(M, \mathbf{e})} = \llbracket [\mathbf{a}]V \vdash r \rrbracket_{(M, \mathbf{e})} : \llbracket [\mathbf{a}]V \rrbracket(M) \rightarrow M$$

where

$$\llbracket [\mathbf{a}]V \rrbracket(M) = \left(\prod_{x \in |V|} [\mathbb{A}^{\#V(x)}, M] \right) \# \mathbb{A}^{\#n}$$

and where $\llbracket [\mathbf{a}]V \vdash t \rrbracket_{(M, \mathbf{e})}$ is inductively defined as follows:

- $\llbracket [\mathbf{a}]V \vdash x(\mathbf{b}) \rrbracket_{(M, \mathbf{e})}(\{ \langle \mathbf{c}_x \rangle m_x \}_{x \in |V|}, \mathbf{d}) = (\mathbf{c}_x \mathbf{c}) \cdot m_x$ for $\mathbf{c} = (\mathbf{a} \mathbf{d}) \cdot \mathbf{b}$,
- $\llbracket [\mathbf{a}]V \vdash \mathbf{o}(t_1, \dots, t_k) \rrbracket_{(M, \mathbf{e})}(\{ \langle \mathbf{c}_x \rangle m_x \}_{x \in |V|}, \mathbf{d}) = \mathbf{e}_k(\mathbf{o}', t'_1, \dots, t'_k)$ for $\mathbf{e}_k : \Sigma(k) \times M^k \rightarrow M$ the k -component of the structure map \mathbf{e} , and

$$\mathbf{o}' = (\mathbf{a} \mathbf{d}) \cdot \mathbf{o} \quad , \quad t'_i = \llbracket [\mathbf{a}]V \vdash t_i \rrbracket_{(M, \mathbf{e})}(\{ \langle \mathbf{c}_x \rangle m_x \}_{x \in |V|}, \mathbf{d}) \quad .$$

The category $\mathbb{T}\text{-Alg}$ is the full subcategory of $F_{\Sigma}\text{-Alg}$ consisting of \mathbb{T} -algebras. $\mathbb{T}\text{-Alg}$ and $\underline{\mathbb{T}}\text{-Alg}$ are isomorphic by construction.

$$\begin{array}{c}
\text{Eqvar} \frac{[\mathbf{a}^n]V \vdash_E t \equiv t'}{[\mathbf{b}^n]V \vdash_E (\mathbf{a}^n \mathbf{b}^n) \cdot t \equiv (\mathbf{a}^n \mathbf{b}^n) \cdot t'} \\
\text{Ref} \frac{[\mathbf{a}]V \vdash_E t}{[\mathbf{a}]V \vdash_E t \equiv t} \quad \text{Sym} \frac{[\mathbf{a}]V \vdash_E t \equiv t'}{[\mathbf{a}]V \vdash_E t' \equiv t} \quad \text{Trans} \frac{[\mathbf{a}]V \vdash_E t \equiv t' \quad [\mathbf{a}]V \vdash_E t' \equiv t''}{[\mathbf{a}]V \vdash_E t \equiv t''} \\
\text{Axiom} \frac{([\mathbf{a}]V \vdash l \equiv r) \in E}{[\mathbf{a}]V \vdash_E l \equiv r} \quad \text{Elim} \frac{[\mathbf{a}, \mathbf{b}]V \vdash_E t \equiv t'}{[\mathbf{a}]V \vdash_E t \equiv t'} (\mathbf{b} \# \mathbf{a}, t, t') \\
\text{Intro} \frac{[\mathbf{a}]V \vdash_E t \equiv t'}{[\mathbf{a}, \mathbf{b}^m]V^{(m)} \vdash_E t\{x(\mathbf{b}_x) \mapsto x(\mathbf{b}_x, \mathbf{b})\}_{x \in |V|} \equiv t'\{x(\mathbf{b}_x) \mapsto x(\mathbf{b}_x, \mathbf{b})\}_{x \in |V|}} \\
\text{with } \mathbf{b} \# \mathbf{a} \text{ and } \forall_{x \in |V|} \mathbf{b} \# \mathbf{b}_x \\
\text{where } |V^{(m)}| = |V| \text{ and } \forall_{x \in |V|} V^{(m)}(x) = V(x) + m \\
\text{Subst}_{\Pi} \frac{[\mathbf{a}]V \vdash_E t \equiv t' \quad [\mathbf{b}_x^{V(x)}]U \vdash_E s_x \equiv s'_x (x \in |V|)}{[\mathbf{a}]U \vdash_E t\{x(\mathbf{b}_x) \mapsto s_x\}_{x \in |V|} \equiv t'\{x(\mathbf{b}_x) \mapsto s'_x\}_{x \in |V|}}
\end{array}$$

Figure 1: Rules of SNEL.

Example 5.3 (continued from Example 5.2). A \mathbb{T}_λ -algebra has a carrier $M \in \mathbf{Nom}$ with structure maps

$$\llbracket \mathbf{V} \rrbracket : \mathbb{A} \rightarrow M, \quad \llbracket \mathbf{L} \rrbracket : \mathbb{A} \times M \rightarrow M, \quad \llbracket \mathbf{A} \rrbracket : M^2 \rightarrow M$$

satisfying the equations of the theory. For instance, according to the equation (α) , we have that

$$\llbracket \mathbf{L} \rrbracket(a, (c a) \cdot m) = \llbracket \mathbf{L} \rrbracket(b, (c b) \cdot m) \quad \text{for all } (\langle c \rangle m, (a, b)) \in [\mathbb{A}, M] \# \mathbb{A}^{\#2}$$

and, according to the equation (η) , we have that

$$\llbracket \mathbf{L} \rrbracket(a, \llbracket \mathbf{A} \rrbracket(m, \llbracket \mathbf{V} \rrbracket(a))) = m \quad \text{for all } (m, a) \in M \# \mathbb{A}.$$

5.5. Synthetic Nominal Equational Logic. For a NEP $\mathbb{T} = (\Sigma, E)$, we consider the EML associated to the NES $\underline{\mathbb{T}}$ in syntactic form, and thereby synthesise a deductive system for deriving valid nominal equations in \mathbb{T} -algebras. The resulting *Synthetic Nominal Equational Logic (SNEL)* has the inference rules given in Figure 1. The *substitution operation* in the rules **Intro** and **Subst $_{\Pi}$** maps

$$t \in T_{\Sigma} \underline{U}, \quad \{ \langle \mathbf{c}_x \rangle s_x \in [\mathbb{A}^{\#U(x)}, T_{\Sigma} X] \}_{x \in |U|}$$

to the nominal term

$$t\{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |U|} \in T_{\Sigma} X$$

defined by structural induction on t as follows:

$$\begin{aligned}
x(\mathbf{a})\{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |U|} &= (\mathbf{c}_x \mathbf{a}) \cdot s_x \\
\mathfrak{o}(t_1, \dots, t_k)\{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |U|} &= \mathfrak{o}(t_1\{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |U|}, \dots, t_k\{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |U|}) \cdot
\end{aligned}$$

Remark 5.4. Note that under the rule Ref, the rules Intro and Subst_{II} are inter-derivable with the rule

$$\text{IntroSubst}_{\text{II}} \frac{[\mathbf{a}]V \vdash_E t \equiv t' \quad [\mathbf{b}_x^{V(x)}, \mathbf{b}]U \vdash_E s_x \equiv s'_x \quad (x \in |V|)}{[\mathbf{a}, \mathbf{b}]U \vdash_E t\{x(\mathbf{b}_x) \mapsto s_x\}_{x \in |V|} \equiv t'\{x(\mathbf{b}_x) \mapsto s'_x\}_{x \in |V|}} (\mathbf{b} \# \mathbf{a})$$

Indeed, the above arises from the rule Intro for \mathbf{b} on the judgement $[\mathbf{a}]V \vdash_E t \equiv t'$ followed by the rule Subst_{II} with respect to the family $[\mathbf{b}_x^{V(x)}, \mathbf{b}]U \vdash_E s_x \equiv s'_x \quad (x \in |V|)$; whilst, conversely, the rule Subst_{II} is the special case of the rule IntroSubst_{II} for \mathbf{b} the empty tuple and the rule Intro arises by instantiating the rule IntroSubst_{II} with the family $[\mathbf{b}_x^{V(x)}, \mathbf{b}^m]V^{(m)} \vdash_E x(\mathbf{b}_x, \mathbf{b}) \equiv x(\mathbf{b}_x, \mathbf{b}) \quad (x \in |V|)$.

We also note that the rule Elim is in fact reversible, as the instantiation of the rule IntroSubst_{II} with the family $[\mathbf{b}_x^{V(x)}, \mathbf{b}]V \vdash_E x(\mathbf{b}_x) \equiv x(\mathbf{b}_x) \quad (x \in |V|)$ yields the derivability of the rule

$$\text{Inc} \frac{[\mathbf{a}]V \vdash_E t \equiv t'}{[\mathbf{a}, \mathbf{b}]V \vdash_E t \equiv t'} (\mathbf{b} \# \mathbf{a})$$

Example 5.5 (continued from Example 5.3). We give a derivation of

$$[\mathbf{a}] x : 1, y : 0 \vdash \mathbf{A}(\mathbf{L}_a \cdot \mathbf{L}_a \cdot x(a), y) \equiv \mathbf{L}_a \cdot x(a)$$

in the SNEL of \mathbb{T}_λ :

$$\begin{aligned}
& \begin{array}{c} [\mathbf{a}, \mathbf{b}] x : 1 \vdash \mathbf{L}_a \cdot x(a) \equiv \mathbf{L}_b \cdot x(b) \quad \text{by Axiom } (\alpha) \\ x \mapsto [\mathbf{c}] x : 1, y : 0 \vdash x(\mathbf{c}) \equiv x(\mathbf{c}) \quad \text{by Ref} \\ \mathbf{A} : \frac{}{[\mathbf{a}, \mathbf{b}] x : 1, y : 0 \vdash \mathbf{L}_a \cdot x(a) \equiv \mathbf{L}_b \cdot x(b)} \text{by Subst}_{\text{II}} \end{array} \\
& \begin{array}{c} [\mathbf{a}, \mathbf{b}] z : 2, w : 0 \vdash \mathbf{A}(\mathbf{L}_a \cdot z(a, b), w) \equiv \mathbf{A}(\mathbf{L}_a \cdot z(a, b), w) \quad \text{by Ref} \\ z \mapsto [\mathbf{a}, \mathbf{b}] x : 1, y : 0 \vdash \mathbf{L}_a \cdot x(a) \equiv \mathbf{L}_b \cdot x(b) \quad \text{by } \mathbf{A} \\ w \mapsto [\] x : 1, y : 0 \vdash y \equiv y \quad \text{by Ref} \\ \mathbf{B} : \frac{}{[\mathbf{a}, \mathbf{b}] x : 1, y : 0 \vdash \mathbf{A}(\mathbf{L}_a \cdot \mathbf{L}_a \cdot x(a), y) \equiv \mathbf{A}(\mathbf{L}_a \cdot \mathbf{L}_b \cdot x(b), y)} \text{by Subst}_{\text{II}} \end{array} \\
& \begin{array}{c} [\mathbf{a}] x : 0, y : 1 \vdash \mathbf{A}(\mathbf{L}_a \cdot x, y(a)) \equiv x \quad \text{by Axiom } (\beta_\kappa) \\ \mathbf{C} : \frac{}{[\mathbf{a}, \mathbf{b}] x : 1, y : 2 \vdash \mathbf{A}(\mathbf{L}_a \cdot x(b), y(a, b)) \equiv x(b)} \text{by Intro} \end{array} \\
& \begin{array}{c} [\mathbf{a}, \mathbf{b}] x : 1, y : 2 \vdash \mathbf{A}(\mathbf{L}_a \cdot x(b), y(a, b)) \equiv x(b) \quad \text{by } \mathbf{C} \\ x \mapsto [\mathbf{b}] x : 1, y : 0 \vdash \mathbf{L}_b \cdot x(b) \equiv \mathbf{L}_b \cdot x(b) \quad \text{by Ref} \\ y \mapsto [\mathbf{a}, \mathbf{b}] x : 1, y : 0 \vdash y \equiv y \quad \text{by Ref} \\ \mathbf{D} : \frac{}{[\mathbf{a}, \mathbf{b}] x : 1, y : 0 \vdash \mathbf{A}(\mathbf{L}_a \cdot \mathbf{L}_b \cdot x(b), y) \equiv \mathbf{L}_b \cdot x(b)} \text{by Subst}_{\text{II}} \end{array} \\
& \frac{[\mathbf{a}, \mathbf{b}] x : 1, y : 0 \vdash \mathbf{A}(\mathbf{L}_a \cdot \mathbf{L}_a \cdot x(a), y) \equiv \mathbf{L}_a \cdot x(a) \quad \text{by Trans(Trans(B, D), Sym(A))}}{[\mathbf{a}] x : 1, y : 0 \vdash \mathbf{A}(\mathbf{L}_a \cdot \mathbf{L}_a \cdot x(a), y) \equiv \mathbf{L}_a \cdot x(a)} \text{by Elim}
\end{aligned}$$

5.6. Soundness. By construction, if a nominal equation $[\mathbf{a}]V \vdash_E t \equiv t'$ is derivable in SNEL, then the equation $\underline{E} \vdash \underline{[\mathbf{a}]V} \vdash t \equiv \underline{[\mathbf{a}]V} \vdash t'$ is derivable in EML. We explain why this is so for each rule.

- The SNEL rule **Eqvar** arises from the fact that

$$\underline{[\mathbf{a}^n]V} \vdash s = \underline{[\mathbf{b}^n]V} \vdash (\mathbf{a}^n \mathbf{b}^n) \cdot s : \mathbb{A}^{\#n} \rightarrow T_\Sigma(\underline{V})$$

for all $\mathbf{b}^n \in \mathbb{A}^{\#n}$ and nominal terms $[\mathbf{a}^n]V \vdash s$.¹

- The SNEL rules **Ref**, **Sym**, **Trans**, and **Axiom** directly mimic the corresponding EML rules.
- The SNEL rule **Elim** arises from the EML rule **Local**₁ with respect to the epimorphic projection map $\mathbb{A}^{\#(n+m)} \twoheadrightarrow \mathbb{A}^{\#n}$ sending $(\mathbf{a}^n, \mathbf{b}^m)$ to (\mathbf{a}^n) .
- The SNEL rule **Intro** arises from the EML rule **Ext** extended with the nominal set $\mathbb{A}^{\#m}$. Note that for $[\mathbf{a}]V \vdash s$, one has that $\underline{[\mathbf{a}^n, \mathbf{b}^m]V} \vdash s \{x(\mathbf{c}_x) \mapsto x(\mathbf{c}_x, \mathbf{b})\}_{x \in |V|}$ amounts to the composite

$$\mathbb{A}^{\#(n+m)} \cong \mathbb{A}^{\#m} \# \mathbb{A}^{\#n} \xrightarrow{\langle \mathbb{A}^{\#m} \rangle [\mathbf{a}]V \vdash s} T_\Sigma(\mathbb{A}^{\#m} \# \underline{V}) \cong T_\Sigma(\coprod_{x \in |V|} \mathbb{A}^{\#(V(x)+m)}) .$$

- The SNEL rule **Subst**_{II} arises from the EML rule

$$\text{Subst}_{\text{II}} \frac{[\mathbf{a}]U \vdash t \equiv [\mathbf{a}]U \vdash t' \quad [\mathbf{b}_x]V \vdash s_x \equiv [\mathbf{b}_x]V \vdash s'_x \quad (x \in |U|)}{([\mathbf{a}]U \vdash t) \left\{ \underline{[\mathbf{b}_x]V} \vdash s_x \right\}_{x \in |U|} \equiv ([\mathbf{a}]U \vdash t') \left\{ \underline{[\mathbf{b}_x]V} \vdash s'_x \right\}_{x \in |U|}}$$

noting that $([\mathbf{a}]U \vdash t) \left\{ \underline{[\mathbf{b}_x]V} \vdash s_x \right\}_{x \in |U|} = \underline{[\mathbf{a}]V} \vdash t \{x(\mathbf{b}_x) \mapsto s_x\}_{x \in |U|}$.

Thus, the soundness of SNEL follows from that of EML.

5.7. Completeness. We provide a sound and complete rewriting-style deduction system for NEPs, referred to as *Synthetic Nominal Rewriting (SNR)*, and establish the completeness of SNEL.

For every NEP \mathbb{T} , the associated NES $\underline{\mathbb{T}}$ is inductive. Indeed, using that in **Nom** finite limits commute with filtered colimits and equivariant functions are epimorphic iff their underlying function is surjective, one sees that the endofunctor F_Σ associated to a nominal signature Σ preserves colimits of ω -chains and epimorphisms. Moreover, since for every $n \in \mathbb{N}$, the right adjoint $[\mathbb{A}^{\#n}, -]$ is also a left adjoint, it follows that, for every variable context V , the nominal set \underline{V} is compact and projective.

For a NEP $\mathbb{T} = (\Sigma, E)$, we consider the construction (3.3) for the associated NES $\underline{\mathbb{T}}$. Since the forgetful functor $|-| : \mathbf{Nom} \rightarrow \mathbf{Set}$ creates colimits, we have the following explicit description. For a nominal set X , the nominal set $(T_\Sigma X)_1$ has as underlying set the quotient $|T_\Sigma X| / \approx_1$ under the equivalence relation \approx_1 on $|T_\Sigma X|$ generated by the following rule:

$$\frac{([\mathbf{a}^n]V \vdash l \equiv r) \in E}{((\mathbf{a}^n \mathbf{b}^n) \cdot l) \{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |V|} \approx_1 ((\mathbf{a}^n \mathbf{b}^n) \cdot r) \{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |V|}}$$

for $\mathbf{b} \in \mathbb{A}^{\#n}$ and $\langle \mathbf{c}_x \rangle s_x \in [\mathbb{A}^{\#V(x)}, T_\Sigma X]$ such that $\mathbf{b} \# \langle \mathbf{c}_x \rangle s_x$ for all $x \in |V|$. The action

¹The omission of this rule in the SNEL presented in Fiore and Hur (2008) is an oversight.

$$\text{Ref} \frac{t \in |T_\Sigma X|}{t \approx_E t} \quad \text{Sym} \frac{t \approx_E t'}{t' \approx_E t} \quad \text{Trans} \frac{t \approx_E t' \quad t' \approx_E t''}{t \approx_E t''}$$

$$\text{Inst} \frac{([\mathbf{a}^n]V \vdash l \equiv r) \in E}{((\mathbf{a}^n \mathbf{b}^n) \cdot l) \{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |V|} \approx_E ((\mathbf{a}^n \mathbf{b}^n) \cdot r) \{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |V|}}$$

with $\mathbf{b} \in \mathbb{A}^{\#n}$ and $\langle \mathbf{c}_x \rangle s_x \in [\mathbb{A}^{\#V(x)}, T_\Sigma X]$ such that $\mathbf{b} \# \langle \mathbf{c}_x \rangle s_x$ for all $x \in |V|$

$$\text{Cong} \frac{s_1 \approx_E s'_1, \dots, s_k \approx_E s'_k}{\mathbf{o}(s_1, \dots, s_k) \approx_E \mathbf{o}(s'_1, \dots, s'_k)} \quad (\mathbf{o} \in \Sigma(k))$$

Figure 2: Rules for \approx_E .

of $(T_\Sigma X)_1$ is given by $\pi \cdot [t]_{\approx_1} = [\pi \cdot t]_{\approx_1}$. The equivariant function $q_0 : T_\Sigma X \rightarrow (T_\Sigma X)_1$ maps a nominal term t to its equivalence class $[t]_{\approx_1}$.

The nominal set $(T_\Sigma X)_n$, for $n \geq 2$, has as underlying set the quotient $|T_\Sigma X| / \approx_n$ under the equivalence relation \approx_n on $|T_\Sigma X|$ generated by the following rules:

$$\frac{s \approx_{n-1} s'}{s \approx_n s'} \quad \frac{s_1 \approx_{n-1} s'_1, \dots, s_k \approx_{n-1} s'_k}{\mathbf{o}(s_1, \dots, s_k) \approx_n \mathbf{o}(s'_1, \dots, s'_k)} \quad (\mathbf{o} \in \Sigma(k))$$

The action of $(T_\Sigma X)_n$ is given by $\pi \cdot [t]_{\approx_n} = [\pi \cdot t]_{\approx_n}$. The equivariant function $q_{n-1} : (T_\Sigma X)_{n-1} \rightarrow (T_\Sigma X)_n$ maps $[t]_{\approx_{n-1}}$ to $[t]_{\approx_n}$.

The nominal set $T_{\mathbb{T}}X$, being the colimit of the chain $\{q_n : (T_\Sigma X)_n \rightarrow (T_\Sigma X)_{n+1}\}_{n \geq 0}$, is given by $|T_{\mathbb{T}}X| = |T_\Sigma X| / \approx_E$ with action $\pi \cdot [t]_{\approx_E} = [\pi \cdot t]_{\approx_E}$, for \approx_E the equivalence relation on $|T_\Sigma X|$ specified by the rules of Figure 2. The quotient map $\mathbf{q}_X^{\mathbb{T}} : T_\Sigma X \rightarrow T_{\mathbb{T}}X$ sends a nominal term t to its equivalence class $[t]_{\approx_E}$.

The rules **Inst** and **Cong** of Figure 2 can be merged into a single one to yield a rewriting-style deduction system. Indeed, by an induction on the depth of proof trees, one shows that the relation \approx_E coincides with the equivalence relation $\approx_E^R \subseteq (T_\Sigma X)^2$ generated by the rule

$$\mathbf{C}[(\mathbf{a}^n \mathbf{b}^n) \cdot l] \{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |V|} \approx_E^R \mathbf{C}[(\mathbf{a}^n \mathbf{b}^n) \cdot r] \{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |V|} \quad (5.2)$$

for $([\mathbf{a}^n]V \vdash l \equiv r) \in E$ and $\mathbf{C}[-]$ a context with one hole and possibly with elements from X , and for $\mathbf{b} \in \mathbb{A}^{\#n}$ and $\langle \mathbf{c}_x \rangle s_x \in [\mathbb{A}^{\#V(x)}, T_\Sigma X]$ such that $\mathbf{b} \# \langle \mathbf{c}_x \rangle s_x$ for all $x \in |V|$. Henceforth, the rewriting of nominal terms by the rule (5.2) is referred to as *Synthetic Nominal Rewriting (SNR)*.

Example 5.6 (cf. the derivation given in Example 5.5). We give a derivation of

$$\mathbf{A}(\mathbf{L}_a. \mathbf{L}_a. x(a), y) \approx_E^R \mathbf{L}_a. x(a) \text{ in } T_{\Sigma_\lambda}(x : 1, y : 0)$$

in the SNR of \mathbb{T}_λ :

$$\begin{aligned}
& \mathbf{A}(\mathbf{L}_a. \mathbf{L}_a. x(a), y) \\
& \quad \text{by } (\alpha) : \approx^{\mathbf{R}} \mathbf{A}(\mathbf{L}_a. [\mathbf{L}_a. x(a) \{x(c) \mapsto x(c)\}], y) \\
& \approx^{\mathbf{R}} \mathbf{A}(\mathbf{L}_a. \mathbf{L}_b. x(b), y) \\
& \quad \text{by } (\beta_\kappa) : \approx^{\mathbf{R}} [\mathbf{A}(\mathbf{L}_a. x, y(a)) \{x \mapsto \mathbf{L}_b. x(b); y(a) \mapsto y\}] \\
& \approx^{\mathbf{R}} \mathbf{L}_b. x(b) \\
& \quad \text{by } (\alpha) : \approx^{\mathbf{R}} [\mathbf{L}_b. x(b) \{x(c) \mapsto x(c)\}] \\
& \approx^{\mathbf{R}} \mathbf{L}_a. x(a) .
\end{aligned}$$

The soundness and completeness of SNR is established by means of the internal completeness of the NES \mathbb{T} :

$$\begin{aligned}
& [\mathbf{a}^n]V \vdash s \equiv t \text{ is satisfied by all } \mathbb{T}\text{-algebras} \\
& \iff \mathbb{T}\text{-Alg} \models \underline{[\mathbf{a}^n]V \vdash s} \equiv \underline{[\mathbf{a}^n]V \vdash t} : \mathbb{A}^{\#n} \rightarrow T_{\Sigma}V \\
& \iff \mathbf{q}_{\underline{V}}^{\mathbb{T}} \circ \underline{[\mathbf{a}^n]V \vdash s} = \mathbf{q}_{\underline{V}}^{\mathbb{T}} \circ \underline{[\mathbf{a}^n]V \vdash t} : \mathbb{A}^{\#n} \rightarrow T_{\mathbb{T}}V \\
& \iff \mathbf{q}_{\underline{V}}^{\mathbb{T}}([\mathbf{a}^n]V \vdash s(\mathbf{a}^n)) = \mathbf{q}_{\underline{V}}^{\mathbb{T}}([\mathbf{a}^n]V \vdash t(\mathbf{a}^n)) \text{ in } |T_{\mathbb{T}}V|/\approx_E \\
& \iff [s]_{\approx_E} = [t]_{\approx_E} \text{ in } |T_{\mathbb{T}}V|/\approx_E \\
& \iff s \approx_E t \text{ in } |T_{\mathbb{T}}V| \\
& \iff s \approx_E^{\mathbf{R}} t \text{ in } |T_{\mathbb{T}}V|.
\end{aligned}$$

The completeness of SNEl follows, as for all $t, t' \in (T_{\Sigma}U)_{\mathbf{d}}$ every proof of $t \approx_E t'$ can be turned into a proof of $[\mathbf{d}]U \vdash_E t \equiv t'$ in SNEl. In particular, concerning the rule **Inst**, for every $([\mathbf{a}^n]V \vdash l \equiv r) \in E$, $\mathbf{b} \in \mathbb{A}^{\#n}$, and $s_x \in (T_{\Sigma}U)_{(\mathbf{c}_x^{V(x)}, \mathbf{c})}$ ($x \in |V|$) with $\mathbf{b} \# \mathbf{c}$, one deduces

$$[\mathbf{b}, \mathbf{c}]U \vdash_E t_l \equiv t_r, \text{ for } t_u = ((\mathbf{a} \ \mathbf{b}) \cdot u) \{x(\mathbf{c}_x) \mapsto s_x\}_{x \in |V|},$$

by means of the rules **Axiom**, **Eqvar**, **Ref**, **IntroSubst_{II}**; and subsequently derives $[\mathbf{d}]U \vdash_E t_l \equiv t_r$ by means of the rules **Elim** and/or **Inc** for any \mathbf{d} such that $t_l, t_r \in (T_{\Sigma}U)_{\mathbf{d}}$.

5.8. Related work. Algebraic structure and rewriting in a nominal setting have already been considered in the literature. Gabbay and Mathijssen (2006, 2007) and Clouston and Pitts (2007) introduced an essentially equivalent notion of nominal algebra and provided sound and complete equational logics for them, whilst Fernández et al. (2004) introduced nominal rewriting.

Our SNEl and the *Nominal Equational Logic (NEL)* of Clouston and Pitts (2007) are equivalent. Indeed, see Hur (2010, Section 8.2.6) for a translation between the equality judgements of SNEl and NEL that respects the corresponding satisfaction relations. Thus, by virtue of the associated completeness theorems, SNEl and NEL establish the same theorems under different syntactic formalisms.

The *Nominal Rewriting (NR)* of (Fernández and Gabbay, 2007) appears to be a term-rewriting version of NEL. However, it has the shortcoming of not being complete for nominal equational reasoning (see (Hur, 2010, Section 8.2.7)).

Our approach allows us to also put the *Equational Logic for Binding Terms (ELBT)* of Hamana (2006) in the nominal context. Whereas SNEL arises from an EML on \mathbf{Nom} , ELBT arises from a related EML on the super-category $\mathbf{Set}^{\mathbb{I}}$, for \mathbb{I} the category of finite sets and injections. Crucially, however, the embedding $\mathbf{Nom} \hookrightarrow \mathbf{Set}^{\mathbb{I}}$ does not preserve the epimorphic projection maps $p_{n,m} : \mathbb{A}^{\#(n+m)} \rightarrow \mathbb{A}^{\#n}$ ($n \geq 0, m > 0$). Thus, the only essential difference between SNEL and ELBT is that the latter lacks the rule Elim (which arises from the EML rule Local_1 with respect to $p_{n,m}$) as it is unsound.

CONCLUSION

We have introduced a categorical framework for the synthesis of equational logics. This comprises a general abstract notion of equational presentation together with an equational deduction system that is sound for a canonical model theory. In this context, we have also introduced a mathematical methodology for establishing completeness. This is based on an internal strong completeness result that typically leads, through an analysis of the construction of free algebras, to a characterisation of satisfiability via a rewriting-style deduction system embedded within the equational deduction system.

Two applications of our theory and methodology were presented. They respectively provide a rational reconstruction of Birkhoff’s Equational Logic and a novel nominal logic for reasoning about algebraic structure with name-binding operators. A further major application was given in Fiore and Hur (2010) with the synthesis of *Second-Order Equational Logic*: a deductive system for equational reasoning about languages with variable binding and parameterised metavariables (see also Fiore and Mahmoud (2010)).

The extension of the theory of this paper from the mono-sorted to the multi-sorted setting requires a more involved categorical theory (see Fiore (2008); Fiore and Hur (2008); Hur (2010)). A yet more comprehensive extension for a theory of rewriting modulo equations has also been developed.

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APPENDIX A. PROOF OF THEOREM 2.2

Notation. For $f : V \otimes A \rightarrow B$, we write $\bar{f} : A \rightarrow [V, B]$ for the transpose of f with respect to the adjunction $V \otimes - \dashv [V, -]$ and $\hat{f} : V \rightarrow [A, B]$ for the transpose of f with respect to the adjunction $- \otimes A \dashv [A, -]$.

Proof of Theorem 2.2. We establish the soundness of each EML rule; *i.e.*, that every \mathbb{S} -algebra satisfying the premises of a rule also satisfies the conclusion. The soundness of the rules Ref, Sym, Trans, and Axiom is trivial. To show the soundness of the rule Subst, one uses that, for all $u_1 : C \rightarrow TB$ and $u_2 : B \rightarrow TA$,

$$\llbracket u_1 \{u_2\} \rrbracket_{(X,s)} = \llbracket u_1 \rrbracket_{(X,s)} \circ \overline{\llbracket u_2 \rrbracket_{(X,s)} \otimes C} : [A, X] \otimes C \rightarrow X.$$

To show the soundness of the rule Ext, one uses that, for all $u : C \rightarrow TA$,

$$\llbracket \langle V \rangle u \rrbracket_{(X,s)} = \llbracket u \rrbracket_{(X,s)} \circ (\hat{\mathbf{p}} \otimes C) \circ \tilde{\alpha}_{[V \otimes A, X], V, C}^{-1} : [V \otimes A, X] \otimes (V \otimes C) \rightarrow X,$$

where

$$\mathbf{p} = \left(([V \otimes A, X] \otimes V) \otimes A \xrightarrow{\tilde{\alpha}_{[V \otimes A, X], V, A}} [V \otimes A, X] \otimes (V \otimes A) \xrightarrow{\epsilon_X^{V \otimes A}} X \right).$$

Finally, to show the soundness of the rule Local, one uses that, for all $u : C \rightarrow TA$ and $e : C' \rightarrow C$,

$$\llbracket u \circ e \rrbracket_{(X,s)} = \llbracket u \rrbracket_{(X,s)} \circ ([A, X] \otimes e) : [A, X] \otimes C' \rightarrow X$$

and that, for every jointly epimorphic family $\{e_i : C_i \rightarrow C\}_{i \in I}$, the family $\{[A, X] \otimes e_i : [A, X] \otimes C_i \rightarrow [A, X] \otimes C\}_{i \in I}$ is also jointly epimorphic. \square

APPENDIX B. PROOF OF THEOREM 3.1

We introduce several lemmas before proceeding to prove the theorem.

Notation. For $f : V \otimes A \rightarrow B$, we write $\bar{f} : A \rightarrow [V, B]$ for the transpose of f with respect to the adjunction $V \otimes - \dashv [V, -]$ and $\hat{f} : V \rightarrow [A, B]$ for the transpose of f with respect to the adjunction $- \otimes A \dashv [A, -]$.

Lemma B.1. *Let $\mathbb{S} = (\mathcal{C}, \mathbf{T}, \mathcal{A})$ be a MES. For every \mathbb{S} -algebra $(X, s : TX \rightarrow X)$, the T -algebra $([V, X], s^V : T[V, X] \rightarrow [V, X])$, where s^V is the transpose of*

$$\underline{s}^V = \left(V \otimes T[V, X] \xrightarrow{\text{st}_{V, [V, X]}} T(V \otimes [V, X]) \xrightarrow{T(\epsilon_X^V)} TX \xrightarrow{s} X \right),$$

is an \mathbb{S} -algebra.

Proof. That $([V, X], s^V)$ is an Eilenberg-Moore algebra follows from transposing the following identities:

- (1) $\underline{s}^V \circ (V \otimes \eta_{[V, X]}) = \epsilon_X^V : V \otimes [V, X] \rightarrow X$,
- (2) $\underline{s}^V \circ (V \otimes \mu_{[V, X]}) = \underline{s}^V \circ (V \otimes T(\underline{s}^V)) : V \otimes TT[V, X] \rightarrow X$.

To show that every equation in \mathcal{A} is satisfied in $([V, X], s^V)$, one uses that, for all $w : C \rightarrow TA$,

$$\overline{[w]}_{([V, X], s^V)} = \overline{[w]_{(X, s)} \circ (\widehat{\mathbf{p}} \otimes C) \circ \alpha_{V, [A, [V, X]], C}^{-1}} : [A, [V, X]] \otimes C \rightarrow [V, X]$$

where \mathbf{p} denotes the composite

$$(V \otimes [A, [V, X]]) \otimes A \xrightarrow{\alpha_{V, [A, [V, X]], A}} V \otimes ([A, [V, X]] \otimes A) \xrightarrow{V \otimes \epsilon_{[V, X]}^A} V \otimes [V, X] \xrightarrow{\epsilon_X^V} X . \quad \square$$

Lemma B.2. *For $\mathbb{S} = (\mathcal{C}, \mathbf{T}, \mathcal{A})$ a MES admitting free algebras, the free \mathbb{S} -algebra monad $\mathbf{T}_{\mathbb{S}}$ on \mathcal{C} is strong. The components of the strength $\text{st}^{\mathbb{S}}$ are given by the unique maps such that the following diagram commutes:*

$$\begin{array}{ccc} V \otimes TT_{\mathbb{S}}X & \xrightarrow{\text{st}_{V, T_{\mathbb{S}}X}} & T(V \otimes T_{\mathbb{S}}X) \xrightarrow{T(\text{st}_{V, X}^{\mathbb{S}})} TT_{\mathbb{S}}(V \otimes X) \\ V \otimes \tau_X^{\mathbb{S}} \downarrow & & \downarrow \tau_{V \otimes X}^{\mathbb{S}} \\ V \otimes T_{\mathbb{S}}X & \overset{\exists! \text{st}_{V, X}^{\mathbb{S}}}{\dashrightarrow} & T_{\mathbb{S}}(V \otimes X) \\ V \otimes \eta_X^{\mathbb{S}} \uparrow & \nearrow \eta_{V \otimes X}^{\mathbb{S}} & \\ V \otimes X & & \end{array}$$

Proof. First note that $\overline{\text{st}_{V, X}^{\mathbb{S}}} : T_{\mathbb{S}}X \rightarrow [V, T_{\mathbb{S}}(V \otimes X)]$ is the unique homomorphic extension of $\overline{\eta_{V \otimes X}^{\mathbb{S}}} : X \rightarrow [V, T_{\mathbb{S}}(V \otimes X)]$ with respect to the \mathbb{S} -algebra $([V, T_{\mathbb{S}}(V \otimes X)], (\tau_{V \otimes X}^{\mathbb{S}})^V)$.

The naturality of $\text{st}^{\mathbb{S}}$ follows from the fact that, for all $f : V \rightarrow V'$ and $g : C \rightarrow C'$, the maps

$$\overline{T_{\mathbb{S}}(f \otimes g) \circ \text{st}_{V, X}^{\mathbb{S}}} , \overline{\text{st}_{V', X'}^{\mathbb{S}} \circ (f \otimes T_{\mathbb{S}}(g))} : T_{\mathbb{S}}X \rightarrow [V, T_{\mathbb{S}}(V' \otimes X')]$$

are both an homomorphic extension of $\overline{\eta_{V' \otimes X'}^{\mathbb{S}} \circ (f \otimes g)} : X \rightarrow [V, T_{\mathbb{S}}(V' \otimes X')]$ with respect to the \mathbb{S} -algebra $([V, T_{\mathbb{S}}(V' \otimes X')], (\tau_{V' \otimes X'}^{\mathbb{S}})^V)$.

Three of the four coherence conditions for strength follow from the fact that the maps

$$\overline{T_{\mathbb{S}}(\lambda_X) \circ \text{st}_{I, X}^{\mathbb{S}}} , \overline{\lambda_{T_{\mathbb{S}}X}} : T_{\mathbb{S}}X \rightarrow [I, T_{\mathbb{S}}X]$$

$$\overline{T_{\mathbb{S}}(\alpha_{U, V, X}) \circ \text{st}_{U, V, X}^{\mathbb{S}}} , \overline{\text{st}_{U, V \otimes X}^{\mathbb{S}} \circ (U \otimes \text{st}_{V, X}^{\mathbb{S}}) \circ \alpha_{U, V, T_{\mathbb{S}}X}} : T_{\mathbb{S}}X \rightarrow [U \otimes V, T_{\mathbb{S}}(U \otimes (V \otimes X))]$$

$$\overline{\text{st}_{V, X}^{\mathbb{S}} \circ (V \otimes \mu_X^{\mathbb{S}})} , \overline{\mu_{V \otimes X}^{\mathbb{S}} \circ T_{\mathbb{S}}(\text{st}_{V, X}^{\mathbb{S}}) \circ \text{st}_{V, T_{\mathbb{S}}X}^{\mathbb{S}}} : T_{\mathbb{S}}T_{\mathbb{S}}X \rightarrow [V, T_{\mathbb{S}}(V \otimes X)]$$

are respectively homomorphic extensions of

$$\overline{\eta_X^{\mathbb{S}} \circ \lambda_X} : X \rightarrow [I, T_{\mathbb{S}}X]$$

$$\overline{\eta_{U \otimes (V \otimes X)}^{\mathbb{S}} \circ \alpha_{U, V, X}} : X \rightarrow [U \otimes V, T_{\mathbb{S}}(U \otimes (V \otimes X))]$$

$$\overline{\text{st}_X^{\mathbb{S}}} : T_{\mathbb{S}}X \rightarrow [V, T_{\mathbb{S}}(V \otimes X)]$$

with respect to the \mathbb{S} -algebras $([I, T_{\mathbb{S}}X], (\tau_X^{\mathbb{S}})^I)$, $([U \otimes V, T_{\mathbb{S}}(U \otimes (V \otimes X))], (\tau_{U \otimes (V \otimes X)}^{\mathbb{S}})^{(U \otimes V)})$, $([V, T_{\mathbb{S}}(V \otimes X)], (\tau_{V \otimes X}^{\mathbb{S}})^V)$. The remaining coherence condition is the triangle in the diagram above. \square

Lemma B.3. *Let $\mathbb{S} = (\mathcal{L}, \mathbf{T}, \mathcal{A})$ be a MES admitting free algebras. Then, the quotient natural transformation $\mathbf{q}^{\mathbb{S}} : T \dashrightarrow T_{\mathbb{S}}$ is a strong functor morphism between the strong monads \mathbf{T} and $\mathbf{T}_{\mathbb{S}}$. That is, the following diagram commutes:*

$$\begin{array}{ccc} V \otimes TX & \xrightarrow{V \otimes \mathbf{q}_X^{\mathbb{S}}} & V \otimes T_{\mathbb{S}}X \\ \text{st}_{V,X} \downarrow & & \downarrow \text{st}_{V,X}^{\mathbb{S}} \\ T(V \otimes X) & \xrightarrow{\mathbf{q}_{V \otimes X}^{\mathbb{S}}} & T_{\mathbb{S}}(V \otimes X) \end{array}$$

Proof. The commutativity of the diagram follows from the fact that both

$$\overline{\text{st}_{V,X}^{\mathbb{S}} \circ (V \otimes \mathbf{q}_X^{\mathbb{S}})}, \quad \overline{\mathbf{q}_{V \otimes X}^{\mathbb{S}} \circ \text{st}_{V,X}^{\mathbb{S}}} : TX \rightarrow [V, T_{\mathbb{S}}(V \otimes X)]$$

are an homomorphic extension of $\overline{\eta_{V \otimes X}^{\mathbb{S}}} : X \rightarrow [V, T_{\mathbb{S}}(V \otimes X)]$ with respect to the Eilenberg-Moore algebra $([V, T_{\mathbb{S}}(V \otimes X)], (\tau_{V \otimes X}^{\mathbb{S}})^V)$ for the monad \mathbf{T} . \square

We are now ready to prove the internal completeness theorem.

Proof of Theorem 3.1. We show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

1 \Rightarrow 2. Holds vacuously.

2 \Rightarrow 3. Because, for all $w : C \rightarrow TA$, the map $\mathbf{q}_A^{\mathbb{S}} \circ w : C \rightarrow T_{\mathbb{S}}A$ factors as the composite

$$\llbracket w \rrbracket_{(T_{\mathbb{S}}A, \tau_A^{\mathbb{S}})} \circ (\widehat{\mathbf{p}} \otimes C) \circ \lambda_C^{-1}$$

for $\mathbf{p} = (I \otimes A \xrightarrow{\lambda_A} A \xrightarrow{\eta_A^{\mathbb{S}}} T_{\mathbb{S}}A)$.

3 \Rightarrow 1. Because for every $(X, s : TX \rightarrow X) \in \mathbb{S}\text{-Alg}$ and $w : C \rightarrow TA$, the interpretation map $\llbracket w \rrbracket_{(X,s)} : [A, X] \otimes C \rightarrow X$ factors as the composite

$$s^* \circ T_{\mathbb{S}}(\epsilon_X^A) \circ \text{st}_{[A,X],A}^{\mathbb{S}} \circ ([A, X] \otimes (\mathbf{q}_A^{\mathbb{S}} \circ w))$$

where $s^* : T_{\mathbb{S}}X \rightarrow X$ is the unique homomorphic extension of the identity map on X with respect to the \mathbb{S} -algebra (X, s) . \square