

Algebraic theories in the presence of binding operators, substitution, etc.

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Overview

- First order syntax: modelled by algebras on Set
- First order syntax with bindings: modelled by subst-algebras (binding-algebras) on $Set^{\mathbb{R}}$
- First order syntax with equations: modelled by algebraic theories (lawvere theories, or variety of algebras).
- First order syntax with bindings and equations: equational theories by Kelly and Power (problematic).
- New equational theories.

- 1 **Abstract Syntax with Binding**
 - First order syntax without binding
 - First order syntax with binding
 - Substitution Algebras

- 2 **Equational Theories**
 - Classical approach
 - New view on equations
 - Examples of equations

Definition and Example

Signature for First Order Syntax

$$\mathcal{S} = (O, a)$$

O is a set of operators

a is an arity function from O to \mathbb{N}

Terms built from Signature

$$\text{Term}_V := V$$

$$| f(\text{Term}_V^1, \dots, \text{Term}_V^n)$$

V is a set of variables

$$f \in O, a(f) = n$$

Example: Group

$$O = \{e, i, m\}$$

$$a(e) = 0, a(i) = 1, a(m) = 2$$

$$V = \{x, y, z, \dots\}$$

$$\text{Term}_V = \{x, e, m(x, m(y, z)), m(e, m(i(x), y)), \dots\}$$

Categorical Construction

- Base Category: *Set* of sets and functions.
- Endofunctor: $\Sigma X = \coprod_{f \in O} X^{a(f)}$
- Example: Group $\Sigma X = 1 + X + X^2$

State	Set	Depth	Group
S_0	\emptyset		\emptyset
S_1	$V + \Sigma S_0$	0	x, y, z, \dots, e
S_2	$V + \Sigma S_1$	≤ 1	$x, e, i(e), i(x), m(e, x), \dots$
S_3	$V + \Sigma S_2$	≤ 2	$x, e, i(x), m(e, x), m(x, i(e)), \dots$
\vdots	\vdots	\vdots	\vdots
S_∞	$Term_V$	$< \infty$	$\dots, m(i(i(x)), m(x, i(e))), \dots$

- Free Monad of Σ : $TX = \mu A.X + \Sigma A$ with $X + \Sigma TX \xrightarrow{\cong[\eta_X, \tau_X]} TX$

Definition of Σ -alg

- objects: $(X, s : \Sigma X \rightarrow X)$ where X, s in *Set*
- morphisms: $f : (X, s) \rightarrow (Y, t)$ where $f : X \rightarrow Y$ in *Set* satisfying

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \\ s \downarrow & & \downarrow t \\ X & \xrightarrow{f} & Y \end{array}$$

Example: Group

- objects: $(X, 1 + X + X^2 \xrightarrow{[e,i,m]} X)$
- morphisms: $f : (X, e, i, m) \rightarrow (Y, e', i', m')$ where $f : X \rightarrow Y$

$$\begin{array}{ccc} 1 & \xrightarrow{id} & 1 \\ e \downarrow & & \downarrow e' \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow i' \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} X^2 & \xrightarrow{f^2} & Y^2 \\ m \downarrow & & \downarrow m' \\ X & \xrightarrow{f} & Y \end{array}$$

Free Algebras

Free Algebras and Free monads

$$\begin{array}{ccccc} (TX, \Sigma TX \xrightarrow{\tau_X} TX) & \Sigma\text{-alg} & (X, s) \\ \uparrow F & \left(\begin{array}{c} \uparrow \\ \dashv \\ \downarrow \end{array} \right) u & \downarrow u \\ \underline{X} & \text{Set} & X \end{array}$$

Free monad $T = uF$

Universal property

$$\begin{array}{ccc} \Sigma TX & \xrightarrow{\Sigma(i,r)^\#} & \Sigma Y \\ \tau_X \downarrow & & \downarrow r \\ TX & \xrightarrow{(i,r)^\#} & Y \\ \eta_X \uparrow & \nearrow i & \\ X & & \end{array} \qquad \begin{array}{ccc} \Sigma TTX & \xrightarrow{\Sigma\mu_X} & \Sigma TX \\ \tau_{TX} \downarrow & & \downarrow \tau_X \\ TTX & \xrightarrow{\mu_X} & TX \\ \eta_{TX} \uparrow & \nearrow id & \\ TX & & \end{array}$$

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Definition and Example

Signature for First Order Syntax with Binding

$$\mathcal{S} = (O, a)$$

O is a set of operators

a is an arity function from O to \mathbb{N}^* , i.e. $a(f) = (a_1, \dots, a_k)$

Terms built from Signature

$$\begin{aligned} \text{Term}(\{x_1, \dots, x_n\}) &:= \{x_1, \dots, x_n\} \\ &| f(x_{n+1}, \dots, x_{n+a_1}).\text{Term}(\{x_1, \dots, x_{n+a_1}\}), \\ &\quad \vdots \\ &\quad x_{n+1}, \dots, x_{n+a_k}.\text{Term}(\{x_1, \dots, x_{n+a_k}\}) \\ &\text{where } f \in O, a(f) = (a_1, \dots, a_k) \end{aligned}$$

Example: Untyped λ -Calculus

$$O = \{\text{lam}, \text{app}\}$$

$$a(\text{lam}) = (1), a(\text{app}) = (0, 0)$$

$$\text{Term}(x_1, x_2) = \{x_1, \text{app}(x_1, x_2), \text{lam}(x_3.\text{app}(x_1, \text{app}(x_2, x_3))) \dots\}$$

$$\text{Term}(\{x_1, \dots, x_n\}) \cong \{[t]_\alpha \mid FV(t) \subseteq \{x_1, \dots, x_n\} \text{ in untyped } \lambda\text{-calculus}\}$$

Category of Presheaves

Definitions

\mathbb{F}	objects	finite cardinals, <i>i.e.</i> $\emptyset, \{x_1\}, \{x_1, x_2\}, \dots$ let n be $\{x_1, \dots, x_n\}$
	morphisms	set functions from n to m
$Set^{\mathbb{F}}$	objects	functors from \mathbb{F} to Set
	morphisms	natural transformations between functors

Interpretations

$X \in Set^{\mathbb{F}}$: a language.

$X(n)$: set of terms in context $\{x_1, \dots, x_n\}$, *i.e.* $\{t \mid x_1, \dots, x_n \vdash t\}$

$X(n) \xrightarrow{X(\rho)} X(m)$: $x_1, \dots, x_n \vdash t \mapsto x_1, \dots, x_m \vdash t[x_{\rho(1)}/x_1, \dots, x_{\rho(n)}/x_n]$

Presheaf of variables and Type constructor for context extension

$V \in Set^{\mathbb{F}}$: $V(n) = \{x_1, \dots, x_n\}$, $V(\rho) = \rho$

$\delta : Set^{\mathbb{F}} \rightarrow Set^{\mathbb{F}}$: $(\delta X)(n) = X(n+1)$, $(\delta X)(\rho) = X(\rho+1)$

Categorical Construction

- Base Category: $Set^{\mathbb{R}}$
- Endofunctor: $\Sigma(X) \stackrel{\text{def}}{=} \coprod_{\substack{f \in O \\ a(f) = (a_i)_{1 \leq i \leq k}}} \prod_{1 \leq i \leq k} \delta^{a_i} X$
- Example: λ -calculus $\Sigma X = \delta X + X^2$

State	Set	\emptyset	$\{x_1\}$	$\{x_1, x_2\}$
S_0	\emptyset	\emptyset	\emptyset	\emptyset
S_1	$V + \Sigma S_0$	\emptyset	x_1	x_1, x_2
S_2	$V + \Sigma S_1$	$\lambda x_1.x_1$	$x_1, x_1 x_1,$ $\lambda x_2.x_1, \lambda x_2.x_2$	$x_1, x_2, x_1 x_2,$ $\lambda x_3.x_3, \dots$
S_3	$V + \Sigma S_2$	$(\lambda x_1.x_1)(\lambda x_1.x_1),$ $\lambda x_1.\lambda x_2.x_1, \dots$	$x_1, (x_1 x_1)(\lambda x_2.x_2),$ $\lambda x_2.\lambda x_3.x_3, \dots$	$x_2, (x_1 x_2)(\lambda x_3.x_3),$ $\lambda x_3.\lambda x_4.x_3, \dots$
\vdots	\vdots	\vdots	\vdots	\vdots
S_∞	<i>Term</i>	<i>Term</i> (\emptyset)	<i>Term</i> ($\{x_1\}$)	<i>Term</i> ($\{x_1, x_2\}$)

- Free Monad of Σ : $TX = \mu A.X + \Sigma A$ with $X + \Sigma TX \xrightarrow{\cong[\eta_X, \tau_X]} TX$

Example: λ -calculus

- objects: $(X, \delta X + X^2 \xrightarrow{[lam, app]} X)$
- morphisms: $f : (X, lam, app) \rightarrow (Y, lam', app')$ where $f : X \rightarrow Y$

$$\begin{array}{ccc}
 \delta X & \xrightarrow{\delta f} & \delta Y \\
 lam \downarrow & & \downarrow lam' \\
 X & \xrightarrow{f} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 X^2 & \xrightarrow{f^2} & Y^2 \\
 app \downarrow & & \downarrow app' \\
 X & \xrightarrow{f} & Y
 \end{array}$$

Universality

$$\begin{array}{ccc}
 \delta TV + TV^2 & \xrightarrow{\delta(i, lam, app)^{\#} + (i, lam, app)^{\#2}} & \delta Y + Y^2 \\
 \tau_V \downarrow & & \downarrow [lam, app] \\
 TV & \xrightarrow{(i, lam, app)^{\#}} & Y \\
 \eta_V \uparrow & & \nearrow i \\
 V & &
 \end{array}$$

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Motivations for substitution as an algebraic operation

- We can express operations involving substitution, *e.g.* β equations in λ -calculus.
- It ensures semantics substitution lemma, *i.e.* compositionality of substitution.
- It automatically verifies some structural rules related to substitution, *e.g.* syntactic substitution lemma and admissibility of cut.

Substitutions on concrete terms

$$\sigma_{k,n} : (x_1, \dots, x_k \vdash t); (x_1, \dots, x_n \vdash u_1, \dots, u_k) \\ \longmapsto x_1, \dots, x_n \vdash t[u_1/x_1, \dots, u_k/x_k]$$

Example: λ -calculus

$$\sigma_{k,n}(t[x_1, \dots, x_k]; u_1, \dots, u_k) = \begin{array}{ll} u_i & \text{if } t = x_i \\ | & (\sigma_{k,n}(t_1; u_1, \dots, u_k))(\sigma_{k,n}(t_2; u_1, \dots, u_k)) & \text{if } t = t_1 t_2 \\ | & \lambda x_{n+1}. \sigma_{k+1, n+1}(t_1[x_1, \dots, x_{k+1}]; u_1, \dots, u_k, x_{n+1}) & \text{if } t = \lambda x_{k+1}. t_1[x_1, \dots, x_{k+1}] \end{array}$$

Categorical Construction

- Substitution: Function from Substitutable Pairs to Term
- Substitutable Pair: $(x_1, \dots, x_k \vdash t); (x_1, \dots, x_n \vdash u_1, \dots, u_k)$

Type Constructor for Substitutable Pairs

$$(X \bullet Y)(\{x_1, \dots, x_n\}) \stackrel{\text{def}}{=} \prod_{k \in \mathbb{N}} X(\{x_1, \dots, x_k\}) \times (Y(\{x_1, \dots, x_n\}))^k / \approx$$

$$= \{(t; \vec{u}) \mid k \in \mathbb{N}, t \in X(\{x_1, \dots, x_k\}), \vec{u} \in (Y(\{x_1, \dots, x_n\}))^k\} / \approx$$

where \approx is the equivalence relation generated by \sim

$$(t; u_1, \dots, u_k) \sim (t'; u'_1, \dots, u'_{k'}) \quad \text{iff} \quad \exists r: k \rightarrow k' X(r)(t) = t' \wedge u_i = u'_{r(i)}$$

The tensor $\bullet : \text{Set}^{\mathbb{F}} \times \text{Set}^{\mathbb{F}} \rightarrow \text{Set}^{\mathbb{F}}$ forms a closed monoidal structure with a unit V and suitable natural transformations α, λ, ρ .

Categorical Construction

- $st_{X,e:V \rightarrow Y} : \Sigma X \bullet Y \rightarrow \Sigma(X \bullet Y)$ is a natural transformation between two functors of type $Set^{\mathbb{F}} \times V \downarrow Set^{\mathbb{F}} \rightarrow Set^{\mathbb{F}}$.
- Example: λ -calculus: $st_{X,e:V \rightarrow Y} : (\delta X + X \times X) \bullet Y \cong (\delta X) \bullet Y + (X \times X) \bullet Y \rightarrow \delta(X \bullet Y) + (X \bullet Y) \times (X \bullet Y)$
- $\sigma : X \bullet X \rightarrow X$ is defined in the following way.

$$\begin{array}{ccccc}
 \Sigma TV \bullet TV & \xrightarrow{st_{TV,\eta_V:V \rightarrow TV}} & \Sigma(TV \bullet TV) & \xrightarrow{\Sigma\sigma} & \Sigma TV \\
 \tau \bullet id \downarrow & & & & \downarrow \tau \\
 TV \bullet TV & \xrightarrow{\sigma} & & & TV \\
 \eta_V \bullet id \uparrow & & & & \\
 V \bullet TV & \xrightarrow{\lambda_{TV}} & & &
 \end{array}$$

Category of \mathcal{S} -subst-algebras

- Objects: $(X, e : V \rightarrow X, h : \Sigma X \rightarrow X, sub : X \bullet X \rightarrow X)$ where (X, e, sub) is a monoid in $Set^{\mathbb{F}}$ and (X, h) is a Σ -algebra such that the following diagram commutes.

$$\begin{array}{ccccc} \Sigma X \bullet X & \xrightarrow{st_{X,e:V \rightarrow X}} & \Sigma(X \bullet X) & \xrightarrow{\Sigma(sub)} & \Sigma X \\ \downarrow h \bullet id & & & & \downarrow h \\ X \bullet X & \xrightarrow{sub} & & & X \end{array}$$

- Morphisms: maps of $Set^{\mathbb{F}}$ which are both Σ -algebra and monoid homomorphisms.
- Initial Object: $(TV, \eta_V : V \rightarrow TV, \tau : \Sigma TV \rightarrow TV, \sigma : TV \bullet TV \rightarrow TV)$

Definition (Category of Σ -subst-algebras)

- Monoidal closed category $\mathcal{C} = (\mathcal{C}, \otimes, I)$
- Strong endofunctor Σ with a strength^a st
- Objects: $X = (X, e, h, sub)$, where (X, e, sub) is a monoid in \mathcal{C} and (X, h) is a Σ -algebra such that the following diagram commutes.

$$\begin{array}{ccccc} \Sigma X \bullet X & \xrightarrow{st_{X,X}} & \Sigma(X \bullet X) & \xrightarrow{\Sigma(sub)} & \Sigma X \\ \downarrow h \bullet id & & & & \downarrow h \\ X \bullet X & \xrightarrow{sub} & & & X \end{array}$$

- Morphisms: maps of \mathcal{C} which are both Σ -algebra and monoid homomorphisms.

^aA strength for Σ is a natural transformation of type $\Sigma(X) \otimes Y \rightarrow \Sigma(X \otimes Y)$ satisfying some coherence conditions.

Theorem

If Σ has a free monad T then $(TI, \eta_I, \sigma, \tau)$ is an initial Σ -subst-algebra, where $\tau : \Sigma TI \rightarrow TI$ is the free Σ -algebra of I and σ is given as follows:

$$\sigma : TI \bullet TI \xrightarrow{\bar{st}_{I, TI}} T(I \bullet TI) \xrightarrow{T\lambda_{TI}} TTI \xrightarrow{\mu_I} TI ,$$

where $\bar{st} : TX \bullet Y \rightarrow T(X \bullet Y)$ is the lifting of the strength st of Σ to its free monad T .

Definition (Category of Σ -ptd-subst-algebras)

- Monoidal closed category $\mathcal{C} = (\mathcal{C}, \otimes, I)$
- u -strong endofunctor Σ with a u -strength^a st for the forgetful functor u from $I \downarrow \mathcal{C}$
- Objects: $X = (X, e, h, sub)$, where (X, e, sub) is a monoid in \mathcal{C} and (X, h) is a Σ -algebra such that the following diagram commutes.

$$\begin{array}{ccccc} \Sigma X \bullet X & \xrightarrow{st_{X,e:I \rightarrow X}} & \Sigma(X \bullet X) & \xrightarrow{\Sigma(sub)} & \Sigma X \\ \downarrow h \bullet id & & & & \downarrow h \\ X \bullet X & \xrightarrow{sub} & & & X \end{array}$$

- Morphisms: maps of \mathcal{C} which are both Σ -algebra and monoid homomorphisms.

^aA u -strength for Σ is a natural transformation of type $\Sigma(X) \otimes uY \rightarrow \Sigma(X \otimes uY)$ satisfying some coherence conditions.

Theorem

$(TI, \eta_I, \sigma, \mu_I)$ is an initial Σ -ptd-subst-algebra, where σ is given by the following composite:

$$\begin{aligned} TI \otimes TI &= TI \otimes u(\eta_I : I \rightarrow TI) \xrightarrow{st_I, \eta_I} T(I \otimes u(\eta_I)) = T(I \otimes TI) \\ &\xrightarrow{T\lambda_{TI}} TTI \xrightarrow{\mu_I} TI . \end{aligned}$$

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Motivation for equations

Group Axioms

Given a group algebra (X, e, i, m) ,

- (G1) $\forall x, y, z \in X \quad m(m(x, y), z) = m(x, m(y, z))$
- (G2) $\forall x \in X \quad m(e, x) = x$
- (G3) $\forall x \in X \quad m(i(x), x) = e$

β, η equations for λ -calculus

Given a λ -calculus algebra (X, e, lam, app, sub) ,

- (β -eqn)
 $\forall t \in \delta X(n), u \in X(n) \quad app(lam(x_{n+1}.t), u) = sub(t; (x_1, \dots, x_n, u))$
- (η -eqn)
 $\forall t \in X(n) \quad lam(x_{n+1}.app(t, x_{n+1})) = t$

Kelly & Power equational theories

KP Theory	Groups
Sig $S : \mathcal{C}_{fp} \rightarrow \mathcal{C}$	Sig $S : \mathbb{N} \rightarrow Set$
	$(0, 1, 2) \mapsto (1, 1, 1)$
$\Sigma X = \coprod_{a \in \mathcal{C}_{fp} } \mathcal{C}(a, X) \otimes S(a)$	$\Sigma X = \coprod_{a \in \mathbb{N}} X^a \times S(a) = 1 + X + X^2$
$TX = \mu A.X + \Sigma A$	$TX = \mu A.X + \Sigma A$
$TX \cong$ all terms freely generated from X	$TX \cong$ all group terms freely generated from a variable set X
(Equation)	(Equation)
$D : \mathcal{C}_{fp} \rightarrow \mathcal{C}$ together with	$D : \mathbb{N} \rightarrow Set : (1, 3) \mapsto (2, 1)$
$\forall a \in \mathcal{C}_{fp} \quad D(a) \begin{array}{c} \xrightarrow{E_L(a)} \\ \xrightarrow{E_R(a)} \end{array} T(a)$	$D(1) = 2 \begin{array}{c} \xrightarrow{E_L(1)} \\ \xrightarrow{E_R(1)} \end{array} T(1)$
	$E_L(1)(1) = m(e, x_1), E_R(1)(1) = x_1$
	$E_L(1)(2) = m(i(x_1), x_1), E_R(1)(2) = e$
	$D(3) = 1 \begin{array}{c} \xrightarrow{E_L(3)} \\ \xrightarrow{E_R(3)} \end{array} T(3)$
	$E_L(3)(1) = m(m(x_1, x_2), x_3),$
	$E_R(3)(1) = m(x_1, m(x_2, x_3))$

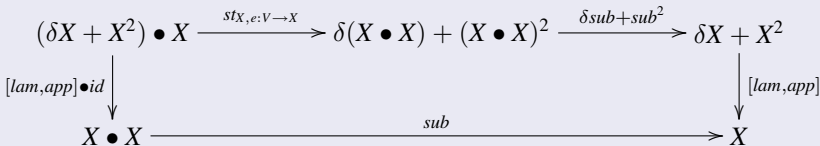
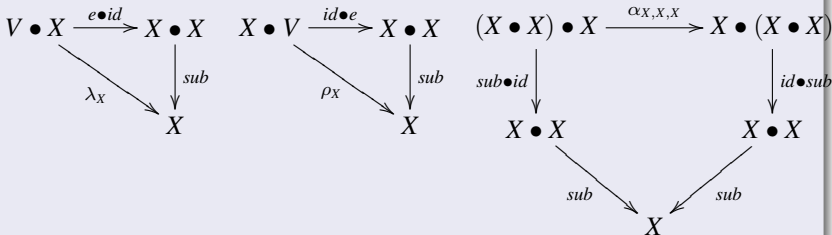
Free Constructions

KP Theory	Groups
<p data-bbox="127 260 648 339">There exists a finitary monad T' such that</p> $T'\text{-Alg} \cong \Sigma\text{-alg}/_{E_L=E_R}$ $F \left(\begin{array}{c} \uparrow \\ \dashv \\ \downarrow \end{array} \right) u$ <p data-bbox="127 619 648 795">where $\Sigma\text{-alg}/_{E_L=E_R}$ is a full subcategory of $\Sigma\text{-alg}$ whose objects are Σ-algebras satisfying the equations.</p>	<p data-bbox="694 260 1215 339">There exists a finitary monad T' such that</p> $T'\text{-Alg} \cong \Sigma\text{-alg}/_{E_L=E_R}$ $F \left(\begin{array}{c} \uparrow \\ \dashv \\ \downarrow \end{array} \right) u$ <p data-bbox="694 619 1215 837">where $\Sigma\text{-alg}/_{E_L=E_R}$ is a full subcategory of $\Sigma\text{-alg}$ whose objects are group-algebras, <i>i.e.</i> (X, e, i, m) satisfying the group axioms $G1$, $G2$, and $G3$</p>

Problems with applying K.P. theory to subst-algebras

Recall: subst-algebras for λ -calculus

Objects: $(X, e : V \rightarrow X, lam : \delta X \rightarrow X, app : X^2 \rightarrow X, sub : X \bullet X \rightarrow X)$
satisfying



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A new view on equational theories: Observation

Example: Group

Given a group algebra $(X, 1 + X + X^2 \xrightarrow{[e,i,m]} X)$,

- (G1) $\forall x, y, z \in X \quad m(m(x, y), z) = m(x, m(y, z))$
- (G2) $\forall x \in X \quad m(e, x) = x$
- (G3) $\forall x \in X \quad m(i(x), x) = e$

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times id} & X \times X \\ \downarrow id \times m & & \downarrow m \\ X \times X & \xrightarrow{m} & X \end{array}$$

G1

$$\begin{array}{ccc} X & \xrightarrow{\langle i, id \rangle} & X \times X \\ \downarrow ! & & \downarrow m \\ 1 & \xrightarrow{e} & X \end{array}$$

G3

$$\begin{array}{ccccc} X & \xrightarrow{\langle !, id \rangle} & 1 \times X & \xrightarrow{e \times id} & X \times X \\ & \searrow id & & & \downarrow m \\ & & & & X \end{array}$$

G2

A new view on equational theories

Observation

$$\begin{array}{ccc} \Sigma\text{-alg} & & D\text{-alg} \\ 1 + X + X^2 & \longmapsto & X^3 + X + X \\ \downarrow [e,i,m] & & \downarrow E_L \quad \downarrow E_R \\ X & & X \end{array}$$

Definition (Equations for algebras)

- a category \mathcal{C} and an endofunctor Σ
- Equation: a domain endofunctor D with functors E_L and E_R from $\Sigma\text{-alg}$ to $D\text{-alg}$ preserving base-objects of algebras, i.e. $u'E_L = u$ and $u'E_R = u$.

$$\begin{array}{ccc} \Sigma\text{-alg} & \begin{array}{c} \xrightarrow{E_L} \\ \xrightarrow{E_R} \end{array} & D\text{-alg} \\ \downarrow u & \swarrow u' & \\ \mathcal{C} & & \end{array}$$

Theorem

If either

- (Cond1) \mathcal{C} is cocomplete and Σ, D preserve colimits of ω -chains.
- (Cond2) \mathcal{C} is cocomplete and well-copowered, and Σ preserves epimorphisms.

then

$$\begin{array}{ccccc} \Sigma\text{-Alg}/_{E_1=E_2}\mathcal{C} & \begin{array}{c} \xleftarrow{L} \\ \perp \\ \xrightarrow{K} \end{array} & \Sigma\text{-Alg} & \begin{array}{c} \xrightarrow{E_1} \\ \xrightarrow{E_2} \end{array} & D\text{-Alg} \\ & \searrow u''=uK & \downarrow u & \swarrow u' & \\ & & \mathcal{C} & & \end{array}$$

Theorem

If either

- (Cond1) \mathcal{C} has coequalizers, pushouts and colimits of chains whose cardinals are less than or equal to a regular cardinal λ , and Σ, D preserve colimits of chains whose cardinals are equal to λ .
- (Cond2) \mathcal{C} has coequalizers, pushouts and colimits of chains (whose cardinals are less than a regular cardinal λ) and is (λ) -well-copowered, and Σ preserves epimorphisms.

then

$$\begin{array}{ccccc} \Sigma\text{-Alg}/_{E_1=E_2}\mathcal{C} & \xleftarrow{L} & \Sigma\text{-Alg} & \xrightarrow{E_1} & D\text{-Alg} \\ & \xrightarrow{K} & \perp & \xrightarrow{E_2} & \\ & & \downarrow u & & \\ & \searrow u''=uK & \mathcal{C} & \swarrow u' & \end{array}$$

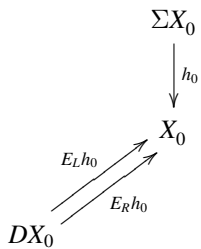
Sketch of the Proof

$$\begin{array}{c} \Sigma X \\ \downarrow h \\ X \end{array}$$

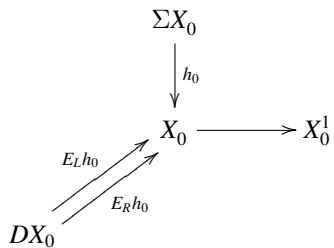
Sketch of the Proof

$$\begin{array}{c} \Sigma X_0 \\ \downarrow h_0 \\ X_0 \end{array}$$

Sketch of the Proof



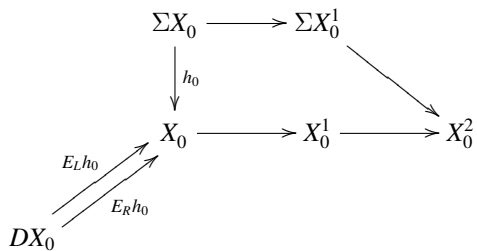
Sketch of the Proof



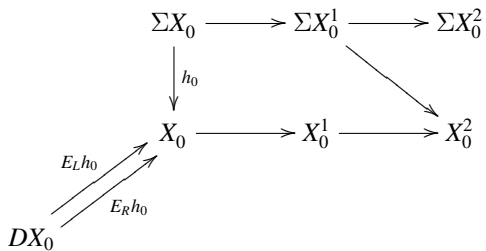
Sketch of the Proof

$$\begin{array}{ccc} \Sigma X_0 & \longrightarrow & \Sigma X_0^1 \\ \downarrow h_0 & & \\ X_0 & \longrightarrow & X_0^1 \\ \nearrow E_L h_0 & & \\ \nearrow E_R h_0 & & \\ DX_0 & & \end{array}$$

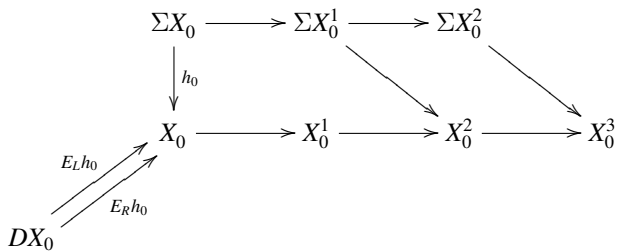
Sketch of the Proof



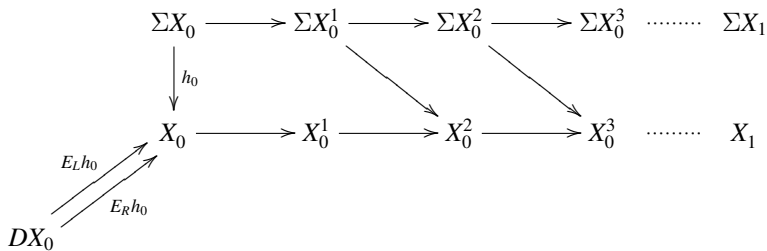
Sketch of the Proof



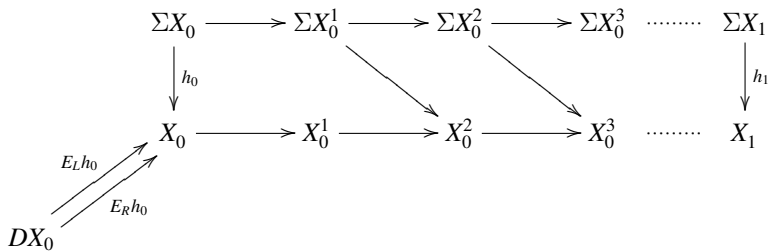
Sketch of the Proof



Sketch of the Proof



Sketch of the Proof



Sketch of the Proof

$$\begin{array}{ccc} \Sigma X_0 & \longrightarrow & \Sigma X_1 \\ \downarrow h_0 & & \downarrow h_1 \\ X_0 & \longrightarrow & X_1 \\ \nearrow E_L h_0 & & \\ \nearrow E_R h_0 & & \\ DX_0 & & \end{array}$$

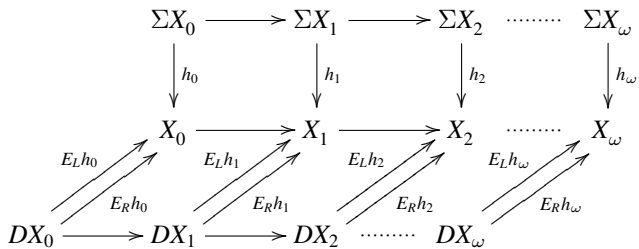
Sketch of the Proof

$$\begin{array}{ccc} \Sigma X_0 & \longrightarrow & \Sigma X_1 \\ \downarrow h_0 & & \downarrow h_1 \\ X_0 & \longrightarrow & X_1 \\ \nearrow E_L h_0 & & \nearrow E_L h_1 \\ DX_0 & \longrightarrow & DX_1 \\ \nwarrow E_R h_0 & & \nwarrow E_R h_1 \end{array}$$

Sketch of the Proof

$$\begin{array}{ccccc} \Sigma X_0 & \longrightarrow & \Sigma X_1 & \longrightarrow & \Sigma X_2 \\ \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 \\ & & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 \\ \nearrow E_L h_0 & & \nearrow E_L h_1 & & \nearrow E_L h_2 \\ DX_0 & \longrightarrow & DX_1 & \longrightarrow & DX_2 \\ \nwarrow E_R h_0 & & \nwarrow E_R h_1 & & \nwarrow E_R h_2 \end{array}$$

Sketch of the Proof



- 1 **Abstract Syntax with Binding**
 - First order syntax without binding
 - First order syntax with binding
 - Substitution Algebras

- 2 **Equational Theories**
 - Classical approach
 - New view on equations
 - Examples of equations

Examples of Equations

subst-algebras for λ -calculus

Objects: $(X, e : V \rightarrow X, lam : \delta X \rightarrow X, app : X^2 \rightarrow X, sub : X \bullet X \rightarrow X)$
satisfying

$$\begin{array}{ccccc}
 V \bullet X & \xrightarrow{e \bullet id} & X \bullet X & & X \bullet V & \xrightarrow{id \bullet e} & X \bullet X & & (X \bullet X) \bullet X & \xrightarrow{\alpha_{X,X,X}} & X \bullet (X \bullet X) \\
 & \searrow \lambda_X & \downarrow sub & & & \searrow \rho_X & \downarrow sub & & \downarrow sub \bullet id & & \downarrow id \bullet sub \\
 & & X & & & & X & & X \bullet X & & X \bullet X \\
 & & & & & & & & \swarrow sub & & \swarrow sub \\
 & & & & & & & & X & &
 \end{array}$$

$$\begin{array}{ccc}
 (\delta X + X^2) \bullet X & \xrightarrow{st_{X,e:V \rightarrow X}} & \delta(X \bullet X) + (X \bullet X)^2 \xrightarrow{\delta sub + sub^2} \delta X + X^2 \\
 \downarrow [lam, app] \bullet id & & \downarrow [lam, app] \\
 X \bullet X & \xrightarrow{sub} & X
 \end{array}$$

Examples of Equations

Example: λ -calculus

(β -eqn)

$$\forall t \in \delta X(n), u \in X(n) \quad \text{app}(\text{lam}(x_{n+1}.t), u) = \text{sub}(t; (x_1, \dots, x_n, u))$$

$$\begin{array}{ccc} \delta X \times X & \xrightarrow{\text{lam} \times \text{id}} & X \times X \\ \text{emb} \downarrow & & \downarrow \text{app} \\ X \bullet X & \xrightarrow{\text{sub}} & X \end{array}$$

(η -eqn)

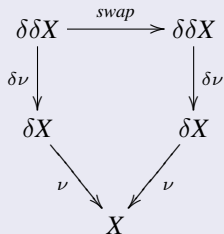
$$\forall t \in X(n) \quad \text{lam}(x_{n+1}.\text{app}(t, x_{n+1})) = t$$

$$\begin{array}{ccccccc} X & \xrightarrow{\langle \text{id}, ! \rangle} & X \times 1 & \xrightarrow{\text{up} \times \lambda e} & \delta X \times \delta X \simeq \delta(X \times X) & \xrightarrow{\delta \text{app}} & \delta X \\ & \searrow \text{id} & & & & & \downarrow \text{lam} \\ & & & & & & X \end{array}$$

Examples of Equations

π -calculus

- Base category $Set^{\mathbb{I}}$
- $\nu : \delta X \rightarrow X$, where $\delta X \cong (V \multimap X)$
- $\nu x.\nu y.t = \nu y.\nu x.t$
- $\forall t \in \delta^2 X(n) \quad \nu x_{n+1}.\nu x_{n+2}.t = \nu x_{n+1}.\nu x_{n+2}.t[x_{n+2}/x_{n+1}, x_{n+1}/x_{n+2}]$



Future Work

- Modelling DILL (Dual Intuitionistic Linear Logic)
- Modelling Dependent type languages.
- Develop the equational theories in pre-ordered setting, and apply them to Term Rewriting Systems.
- Find more applications of the equational theories. Example: the free-algebra models for the π -calculus by Ian Stark.