Term Equational Rewrite Systems and Logics

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Outline of the talk

1. General abstract Definition of Term Equational Rewrite System (TERS).
2. Development of the Theory of Term Equational Rewrite Systems.
3. Applications of Term Equational Rewrite Systems.

Overview: Definition of TERS

Term Equational Rewrite Systems are a framework for developing systems/logics of equations and rewrites.

1. **Equational logic**
   reasoning about equality between terms.
2. **Rewriting system/logic**
   deriving/reasoning about rewrite relation betwen terms.

$\Longrightarrow$

- terms
- equational/rewrite judgements
- algebraic models

Judgements

$$\Gamma \vdash t \equiv t' \qquad \Gamma \vdash t \gg t'$$

Overview: Theory of TERS

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$$\Gamma \vdash t \equiv t' \qquad \Gamma \vdash t \gg t'$$

$$\mathcal{A} \xrightarrow{\text{TERL}} \Gamma \vdash t \star t' \quad (\star \in \{\equiv, \gg\})$$

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$\Downarrow$ Soundness of TERL

$$\forall_{\mathcal{M}} \quad \mathcal{M} \models \mathcal{A} \implies \mathcal{M} \models \Gamma \vdash t \star t'$$

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$\Updownarrow$ Internal Completeness

$$\mathsf{FM}(\mathcal{A}) \models \Gamma \vdash t \star t'$$

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Explicit Construction of $\mathsf{FM}(\mathcal{A})$
(Fiore & Hur, ICALP 07)

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$$\mathcal{A} \xrightarrow{\text{compact logic}} \Gamma \vdash t \star t'$$

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FM($\mathcal{A}$)

(Fiore & Hur, ICALP 07)

e.g., standard term rewriting modulo equations

$$\mathcal{A} \xrightarrow{\text{compact logic}} \Gamma \vdash t \star t'$$

Overview: Applications of TERS

1. Equational logic
   - algebraic theories and first-order equational logic
   - Nominal Equational Logic
     (Gabbay & Mathijssen 06; Clouston & Pitts 07)
2. Rewriting system/logic
   - first-order Term Rewriting System
   - Binding Term Rewriting System (Hamana 03)
   - Nominal Rewriting System (Fernández,Gabbay,Mackie 04)
   - Combinatory Reduction System (Klop 80)

In this talk, in order to convey the basic ideas more easily, we consider TERSs in restricted form only dealing with

- equations (not rewriting)
- single-sorted (not multi-sorted)

Definition: I. Terms and Judgements

- Symmetric monoidal closed category $(\mathscr{D}, I, \otimes, [-,=])$
- Strong monad $\mathbb{T} = (T, \eta, \mu, \tau)$

$$\tau_{X,Y} : X \otimes TY \to T(X \otimes Y)$$

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Universe of discourse

Syntax

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Variables

Terms with variables in V

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Parameter (*e.g.*, Atoms, Object Variables)

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1. Generalised Term $t : U \to TV$, denoted $U, V \vdash t$

   Parameter (*e.g.*, Atoms, Object Variables)

2. Generalised Equation $U, V \vdash t \equiv t'$

Definition: II. Algebraic Models

- Models for $\mathbb{T}$: Eilenberg-Moore algebras for $\mathbb{T}$

$$(D, \xi : TD \to D)$$

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$$\mathscr{D} \circlearrowleft \mathbb{T}$$

- Models for $\mathbb{T}$: Eilenberg-Moore algebras for $\mathbb{T}$

$$(D, \xi : TD \to D)$$

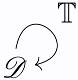
- Satisfaction relation

$$(D,\xi) \;\models\; U, V \vdash t \equiv t'$$

$$\Updownarrow \text{Def}$$

$$[V, D] \otimes U \;\underset{\llbracket t' \rrbracket_{(D,\xi)}}{\overset{\llbracket t \rrbracket_{(D,\xi)}}{\rightrightarrows}}\; D \qquad (\llbracket t \rrbracket_{(D,\xi)} = \llbracket t' \rrbracket_{(D,\xi)})$$

Definition: II. Algebraic Models



- Models for $\mathbb{T}$: Eilenberg-Moore algebras for $\mathbb{T}$

$$(D, \xi : TD \to D)$$

Interpretation of Operators in D

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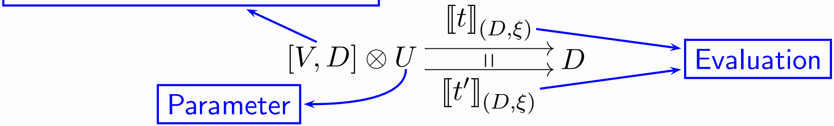

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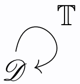

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$\Updownarrow$ Def

Assignment of Variables in D

$$[V, D] \otimes U \underset{[\![t']\!]_{(D,\xi)}}{\overset{[\![t]\!]_{(D,\xi)}}{\rightrightarrows}} D \quad (\text{the two arrows are equal})$$

Parameter

Evaluation

where

$$[\![t]\!]_{(D,\xi)} : [V, D] \otimes U \xrightarrow{id \otimes t} [V, D] \otimes TV \xrightarrow{\tau} T([V, D] \otimes V) \xrightarrow{T\epsilon} TD \xrightarrow{\xi} D$$

Theory: TERL

$$\mathcal{A} \xrightarrow{\text{TERL}} U, V \vdash t \equiv t'$$

Theory: TERL

- Equivalence relation

$$\frac{}{U, V \vdash t \equiv t} \quad \frac{U, V \vdash t \equiv t'}{U, V \vdash t' \equiv t} \quad \frac{U, V \vdash t \equiv t' \quad U, V \vdash t' \equiv t''}{U, V \vdash t \equiv t''}$$

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- Axiom

$$\frac{}{U, V \vdash t \equiv t'} \ (U, V \vdash t \equiv t') \in \mathcal{A}$$

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$$\frac{U, W \vdash t_1 \equiv t_1' \quad W, V \vdash t_2 \equiv t_2'}{U, V \vdash t_1[t_2] \equiv t_1'[t_2']}$$

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- Tensor Extension

$$\frac{U,V \vdash t \equiv t'}{W \otimes U, W \otimes V \vdash \langle W \rangle t \equiv \langle W \rangle t'}$$

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- Local Character

$$\frac{U_i, V \vdash t \circ e_i \equiv t' \circ e_i \quad (i \in I)}{U,V \vdash t \equiv t'}\ \{\, e_i : U_i \to U \,\}_{i \in I} \text{ jointly epi}$$

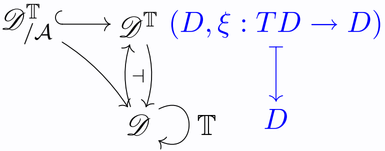
Theory: Soundness of TERL

$$\mathcal{A} \xrightarrow{\text{TERL}} U, V \vdash t \equiv t'$$

$$\Downarrow$$

$$\forall_{(D,\xi)} \quad (D,\xi) \models \mathcal{A} \quad \Longrightarrow \quad (D,\xi) \models U, V \vdash t \equiv t'$$

Theory: Internal Completeness



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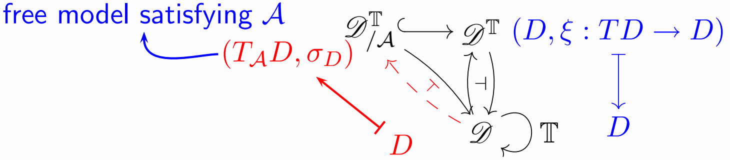

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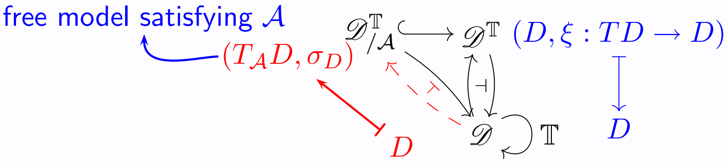
$$\Updownarrow$$

$$(T_{\mathcal{A}}V, \sigma_V) \;\models\; U, V \vdash t \equiv t'$$

Theory: Internal Completeness



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$$\Updownarrow$$

$$(T_{\mathcal{A}}V, \sigma_V) \models U, V \vdash t \equiv t'$$

$$\Updownarrow$$

$$q_V \circ t = q_V \circ t' : U \underset{t'}{\overset{t}{\rightrightarrows}} TV \xrightarrow{q_V} T_{\mathcal{A}}V$$

where

$q_V : TV \to T_{\mathcal{A}}V$ is the quotient of TV under $\mathcal{A}$.

Theory: Construction of q_V

The theory of Equational Systems (Fiore & Hur, ICALP 07) provides an explicit construction of the quotient map

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If

- $\mathscr{D}$ is cocomplete
- The endofunctors T and $[W, -] \otimes R$ for every $(R, W \vdash s \equiv s') \in \mathcal{A}$ preserve colimits of ω-chains and epimorphisms.

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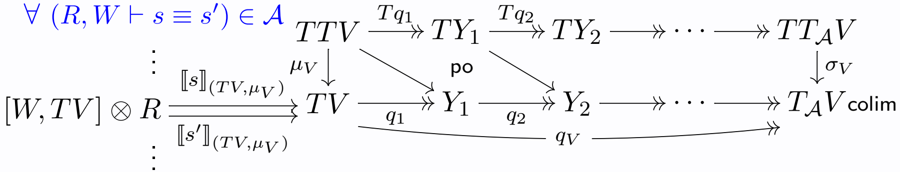

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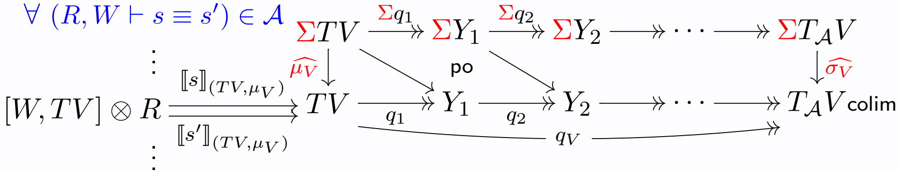
$$q_V : TV \to T_{\mathcal{A}}V \ .$$

If

- $\mathscr{D}$ is cocomplete
- The endofunctors Σ and $[W, -] \otimes R$ for every $(R, W \vdash s \equiv s') \in \mathcal{A}$ preserve colimits of ω-chains and epimorphisms.
- $\mathbb{T}$ is a free monad of an endofunctor Σ.

then



Theory: Towards External Completeness

$$\mathcal{A} \models U, V \vdash t \equiv t'$$

$$\Updownarrow$$

$$q_V \circ t = q_V \circ t' : U \underset{t'}{\overset{t}{\rightrightarrows}} TV \xrightarrow{q_V} T_{\mathcal{A}} V$$

$$+$$

Explicit construction of q_V

$$\Downarrow$$

We may synthesize a sound and complete logic.

Application

Nominal Term Equational System and Logic (NTES & NTEL)

NTES/L: S.M.C. Category

1. $\mathcal{N}om$: the category of nominal sets
2. $(1\,,\,\#\,,\,[-,=])$: symmetric monoidal closed structure

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1. $\mathcal{N}om$: the category of nominal sets
2. $(1\,,\,\#\,,\,[-,=])$: symmetric monoidal closed structure

- 1 : a singleton nominal set
- $M \# M' \triangleq \{ (m, m') \in M \times M' \mid \mathsf{supp}(m) \cap \mathsf{supp}(m') = \emptyset \}$
- $\mathbb{A}$: the nominal set of atoms
- $\mathbb{A}^{\# n} = \{ (a_1, \ldots, a_n) \in \mathbb{A}^n \mid \forall_{1 \le i \ne j \le n}\, a_i \# a_j \}$
- $[\mathbb{A}^{\# n}, M] = \{ \langle \vec{a}^n \rangle\, m \mid (\vec{a}^n) \in \mathbb{A}^{\# n}, m \in M \}$

  where

  $$\vec{a}^n \triangleq a_1, \ldots, a_n$$

  $\langle \vec{a}^n \rangle\, m$ is an α-equivalence class

  e.g. $\langle a, b \rangle\, f(a, b, a) = \langle c, d \rangle\, f(c, d, c)$

NTES/L: Strong Monad

- Signature $\Sigma = \{\, \Sigma(k) \,\}_{k \in \mathbb{N}}$
  for $\Sigma(k)$ a nominal set of operators of arity k

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- Signature $\Sigma = \{\, \Sigma(k) \,\}_{k \in \mathbb{N}}$
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- Endofunctor

$$
\begin{aligned}
F_\Sigma(M) &= \coprod_{k \in \mathbb{N}} \Sigma(k) \times M^k \\
&= \{\, \mathsf{o}\; m_1 \ldots m_k \mid k \in \mathbb{N}, \mathsf{o} \in \Sigma(k), m_1, \ldots, m_k \in M \,\}
\end{aligned}
$$

- with Strength

$$
\begin{array}{rrcl}
\tau : & M' \# F_\Sigma(M) & \to & F_\Sigma(M' \# M) \\
& (m', \mathsf{o}\; m_1 \ldots m_k) & \mapsto & \mathsf{o}\; (m', m_1) \ldots (m', m_k)
\end{array}
$$

NTES/L: Strong Monad

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  for $\Sigma(k)$ a nominal set of operators of arity k
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- with Strength

$$\begin{array}{rrcl} \tau : & M' \# F_\Sigma(M) & \rightarrow & F_\Sigma(M' \# M) \\ & (m', \mathsf{o}\ m_1 \ldots m_k) & \mapsto & \mathsf{o}\ (m', m_1) \ldots (m', m_k) \end{array}$$

- Free monad $\mathbb{T}_\Sigma = (T_\Sigma, \eta, \mu)$

$$\begin{array}{rcll} t \in T_\Sigma M & ::= & m & (\, m \in M \,) \\ & \mid & \mathsf{o}\ t_1 \ldots t_k & (\, \mathsf{o} \in \Sigma(k), t_1, .., t_k \in T_\Sigma M \,) \end{array}$$

- with Inductively defined Strength

NTES/L: Terms and Judgements

Generalised terms $U \to T_\Sigma V$

NTES/L: Terms and Judgements

Generalised terms $U \to T_\Sigma V$

Choose

$$U = \mathbb{A}^{\#n} \qquad \text{for } n \in \mathbb{N}$$

$$V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_k} \qquad \text{for } k \in \mathbb{N}, n_1, \ldots, n_k \in \mathbb{N}$$

Generalised terms $U \to T_\Sigma V$

Choose

$$U = \mathbb{A}^{\# n} \qquad \text{for } n \in \mathbb{N}$$

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Then

$$T_\Sigma V$$

$$x_1 : n_1, \ldots, x_k : n_k \vdash t$$

Inductively defined by

$$\frac{}{x_1 : n_1, \ldots, x_k : n_k \vdash x_i(a'_1, \ldots, a'_{n_i})} \left[\begin{array}{l} i \in \{1, \ldots, k\}, \\ (\vec{a'}^{n_i}) \in \mathbb{A}^{\# n_i} \end{array} \right]$$

$$\frac{x_1 : n_1, \ldots, x_k : n_k \vdash t_i \quad (1 \leq i \leq k)}{x_1 : n_1, \ldots, x_k : n_k \vdash \mathsf{o}\ t_1 \ldots t_k} \left[\mathsf{o} \in \Sigma(k) \right]$$

NTES/L: Terms and Judgements

Generalised terms $U \to T_\Sigma V$

Choose

$$U = \mathbb{A}^{\#n} \qquad \text{for } n \in \mathbb{N}$$

$$V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_k} \qquad \text{for } k \in \mathbb{N}, n_1, \ldots, n_k \in \mathbb{N}$$

Then

$$U \to T_\Sigma V$$
$$\langle \vec{a}^n \rangle \, x_1 : n_1, \ldots, x_k : n_k \vdash t$$

Inductively defined by

$$\frac{}{\langle \vec{a}^n \rangle \, x_1 : n_1, \ldots, x_k : n_k \vdash x_i(a'_1, \ldots, a'_{n_i})} \left[\begin{array}{l} i \in \{1, \ldots, k\}, \\ (\vec{a'}^{n_i}) \in \mathbb{A}^{\#n_i}, \ \{\vec{a'}^{n_i}\} \subseteq \{\vec{a}^n\} \end{array} \right]$$

$$\frac{\langle \vec{a}^n \rangle \, x_1 : n_1, \ldots, x_k : n_k \vdash t_i \quad (1 \leq i \leq k)}{\langle \vec{a}^n \rangle \, x_1 : n_1, \ldots, x_k : n_k \vdash \mathsf{o} \, t_1 \ldots t_k} \left[\begin{array}{l} \mathsf{o} \in \Sigma(k), \\ \mathsf{supp}(\mathsf{o}) \subseteq \{a_1, \ldots, a_n\} \end{array} \right]$$

Generalised terms $U \to T_\Sigma V$

Choose

$$U = \mathbb{A}^{\#n} \qquad \text{for } n \in \mathbb{N}$$

$$V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_k} \qquad \text{for } k \in \mathbb{N}, n_1, \ldots, n_k \in \mathbb{N}$$

Then

Generalised terms & judgements

$$\langle \vec{a}^n \rangle\, x_1 : n_1, \ldots, x_k : n_k \vdash t$$

$$\langle \vec{a}^n \rangle\, x_1 : n_1, \ldots, x_k : n_k \vdash t \equiv t'$$

Inductively defined by

$$\frac{}{\langle \vec{a}^n \rangle\, x_1 : n_1, \ldots, x_k : n_k \vdash x_i(a'_1, \ldots, a'_{n_i})} \left[\begin{array}{l} i \in \{1, \ldots, k\}, \\ (\vec{a'}^{n_i}) \in \mathbb{A}^{\#n_i},\ \{\vec{a'}^{n_i}\} \subseteq \{\vec{a}^n\} \end{array}\right]$$

$$\frac{\langle \vec{a}^n \rangle\, x_1 : n_1, \ldots, x_k : n_k \vdash t_i \quad (1 \leq i \leq k)}{\langle \vec{a}^n \rangle\, x_1 : n_1, \ldots, x_k : n_k \vdash \mathsf{o}\ t_1 \ldots t_k} \left[\begin{array}{l} \mathsf{o} \in \Sigma(k), \\ \mathsf{supp}(\mathsf{o}) \subseteq \{a_1, \ldots, a_n\} \end{array}\right]$$

NTES/L: Algebraic Models

- Algebras

$$\text{Eilenberg-Moore algebras for } \mathbb{T}_\Sigma$$
$$||$$
$$F_\Sigma\text{-algebras } (M, \mathsf{e})$$

where

$$M \in \mathcal{N}om$$
$$\mathsf{e} = \{\, \mathsf{e}_n : \Sigma(n) \times M \to M \,\}_{n \in \mathbb{N}}$$

NTES/L: Algebraic Models

- Algebras

$$\text{Eilenberg-Moore algebras for } \mathbb{T}_\Sigma$$
$$||$$
$$F_\Sigma\text{-algebras } (M, \mathsf{e})$$

where

$$M \in \mathcal{N}om$$
$$\mathsf{e} = \{\, \mathsf{e}_n : \Sigma(n) \times M \to M \,\}_{n \in \mathbb{N}}$$

- Satisfaction relation

$$(M, \mathsf{e}) \models \langle \vec{a}^n \rangle\, x_1 : n_1, \ldots, x_k : n_k \vdash t \equiv t'$$

$$\Updownarrow \text{ Def}$$

$$[\mathbb{A}^{\#n_1}, M] \times \ldots \times [\mathbb{A}^{\#n_k}, M] \times \mathbb{A}^{\#n} \underset{[\![t']\!]_{M,\mathsf{e}}}{\overset{[\![t]\!]_{M,\mathsf{e}}}{\rightrightarrows}} M \quad (=)$$

NTES/L: Untyped λ-calculus (Example)

- Signature

$$\begin{array}{ll} \Sigma_\lambda(0) = \{V_a \mid a \in \mathbb{A}\} & \pi \cdot V_a = V_{\pi(a)} \\ \Sigma_\lambda(1) = \{L_a \mid a \in \mathbb{A}\} & \pi \cdot L_a = L_{\pi(a)} \\ \Sigma_\lambda(2) = \{A\} & \pi \cdot A = A \end{array}$$

- Axioms

$$\begin{array}{rll} \langle a, b\rangle\, x : 1 & \vdash L_a\, x(a) \equiv L_b\, x(b) & (\alpha) \\ \langle a\rangle\, x : 0 & \vdash L_a\, (A\, x()\, V_a) \equiv x() & (\eta) \\ & \vdots & \end{array}$$

- Algebras

$$\begin{array}{l} [\![V]\!] : \mathbb{A} \to M \\ [\![L]\!] : \mathbb{A} \times M \to M \\ [\![A]\!] : M^2 \to M \end{array}$$

- Signature

$$\begin{array}{ll} \Sigma_\lambda(0) = \{V_a \mid a \in \mathbb{A}\} & \pi \cdot V_a = V_{\pi(a)} \\ \Sigma_\lambda(1) = \{L_a \mid a \in \mathbb{A}\} & \pi \cdot L_a = L_{\pi(a)} \\ \Sigma_\lambda(2) = \{A\} & \pi \cdot A = A \end{array}$$

- Axioms

$$\begin{array}{rll} \langle a, b\rangle\, x : 1 & \vdash L_a\, x(a) \equiv L_b\, x(b) & (\alpha) \\ \langle a\rangle\, x : 0 & \vdash L_a\, (A\, x()\, V_a) \equiv x() & (\eta) \\ & \vdots & \end{array}$$

- Algebras

$$\begin{array}{l} [\![V]\!] : \mathbb{A} \to M \\ [\![L]\!] : \mathbb{A} \times M \to M \\ [\![A]\!] : M^2 \to M \end{array}$$

$$[\![L]\!] : [\mathbb{A}, M] \to M$$

NTES/L: Sound Logic NTEL

Ref Sym Trans Axiom

NTES/L: Sound Logic NTEL

Ref   Sym   Trans   Axiom

$$\text{Subst}\ \frac{\langle\vec{a}^n\rangle\,\Delta \vdash t \equiv t' \qquad \{\, \langle\vec{b_x}^{\Delta(x)}\rangle\,\Gamma \vdash s_x \equiv s'_x \,\}_{x\in|\Delta|}}{\langle\vec{a}^n\rangle\,\Gamma \vdash t[x \mapsto \langle\vec{b_x}\rangle\, s_x]_{x\in|\Delta|} \;\equiv\; t'[x \mapsto \langle\vec{b_x}\rangle\, s'_x]_{x\in|\Delta|}}$$

Ref Sym Trans Axiom

$$\text{Subst}\ \frac{\langle \vec{a}^n \rangle\, \Delta \vdash t \equiv t' \qquad \{\, \langle \vec{b_x}^{\Delta(x)} \rangle\, \Gamma \vdash s_x \equiv s'_x \,\}_{x \in |\Delta|}}{\langle \vec{a}^n \rangle\, \Gamma \vdash t[x \mapsto \langle \vec{b_x} \rangle\, s_x]_{x \in |\Delta|} \;\equiv\; t'[x \mapsto \langle \vec{b_x} \rangle\, s'_x]_{x \in |\Delta|}}$$

Tensor Extension

$$\text{Intro}\ \frac{\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t' \qquad \vec{b}^m \# \vec{a}^n \qquad \{\, (\vec{c_x}^{\Gamma(x)}) \,\}_{x \in |\Gamma|} \# \vec{b}^m}{\langle \vec{a}^n, \vec{b}^m \rangle\, \Gamma^{[m]} \vdash t[x \mapsto \langle \vec{c_x} \rangle\, x(\vec{c_x}, \vec{b})]_x \equiv t'[x \mapsto \langle \vec{c_x} \rangle\, x(\vec{c_x}, \vec{b})]_x}$$

where $|\Gamma^{[m]}| = |\Gamma|$ and $\forall_{x \in |\Gamma|}\ \Gamma^{[m]}(x) = \Gamma(x) + m$

NTES/L: Sound Logic NTEL

Ref   Sym   Trans   Axiom

$$\text{Subst}\ \frac{\langle \vec{a}^n \rangle\, \Delta \vdash t \equiv t' \qquad \{\, \langle \vec{b_x}^{\Delta(x)} \rangle\, \Gamma \vdash s_x \equiv s'_x \,\}_{x \in |\Delta|}}{\langle \vec{a}^n \rangle\, \Gamma \vdash t[x \mapsto \langle \vec{b_x} \rangle\, s_x]_{x \in |\Delta|} \;\equiv\; t'[x \mapsto \langle \vec{b_x} \rangle\, s'_x]_{x \in |\Delta|}}$$

Tensor Extension

$$\text{Intro}\ \frac{\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t' \qquad \vec{b}^m \,\#\, \vec{a}^n \qquad \{\, (\vec{c_x}^{\Gamma(x)}) \,\}_{x \in |\Gamma|} \,\#\, \vec{b}^m}{\langle \vec{a}^n, \vec{b}^m \rangle\, \Gamma^{[m]} \vdash t[x \mapsto \langle \vec{c_x} \rangle\, x(\vec{c_x}, \vec{b})]_x \;\equiv\; t'[x \mapsto \langle \vec{c_x} \rangle\, x(\vec{c_x}, \vec{b})]_x}$$

where $|\Gamma^{[m]}| \;=\; |\Gamma|$ and $\forall_{x \in |\Gamma|}\ \Gamma^{[m]}(x) \;=\; \Gamma(x) + m$

Local Character

$$\text{Elim}\ \frac{\langle \vec{a}^n, \vec{b}^m \rangle\, \Gamma \vdash t \equiv t'}{\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t'}\ \vec{b} \,\#\, \vec{a}, t, t'$$

NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v \Longleftrightarrow \mathbb{A}^{\#m} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma_\mathcal{A}} V$$

where $V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_l}$

NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma_\mathcal{A}} V$$

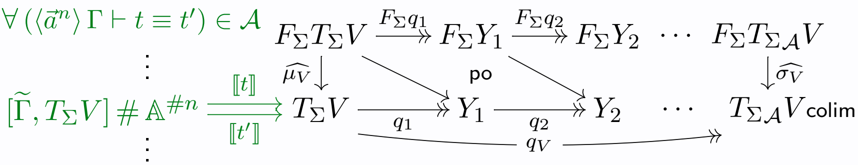
where $V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_l}$

$$T_\Sigma V = \{\, x_1 : n_1, \ldots, x_l : n_l \vdash t \,\}$$

$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma\mathcal{A}} V$$

where $V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_l}$

$$\boxed{T_\Sigma V = \{\, x_1 : n_1, \ldots, x_l : n_l \vdash t \,\}}$$



$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma\mathcal{A}} V$$

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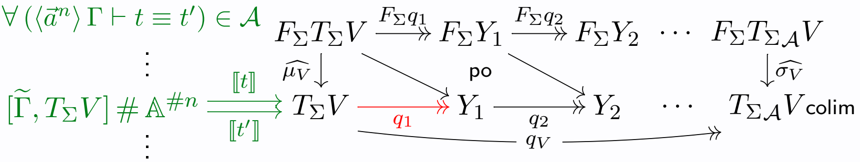
$$T_\Sigma V = \{\, x_1 : n_1, \ldots, x_l : n_l \vdash t \,\}$$



$$\text{Ref}^1\ \frac{}{t \equiv^1 t} \qquad \text{Sym}^1\ \frac{t \equiv^1 t'}{t' \equiv^1 t} \qquad \text{Trans}^1\ \frac{t \equiv^1 t' \quad t' \equiv^1 t''}{t' \equiv^1 t}$$

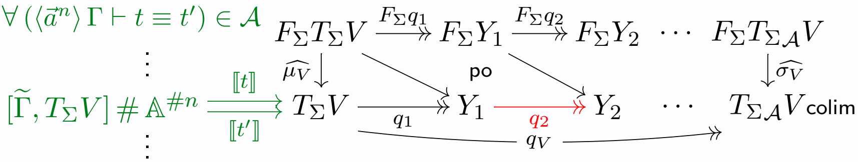
$$\text{Axiom}^1\ \frac{(\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t') \in \mathcal{A} \qquad \vec{b}^n \# \{\, \langle \vec{c_x} \rangle\, s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_\Sigma V] \,\}_{x \in |\Gamma|}}{((\vec{a}\,\vec{b}) \cdot t)[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x \equiv^1 ((\vec{a}\,\vec{b}) \cdot t')[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x}$$

NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma\mathcal{A}} V$$

where $V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_l}$

$$T_\Sigma V = \{\, x_1 : n_1, \ldots, x_l : n_l \vdash t \,\}$$



$$\text{Ref}^2\ \frac{}{t \equiv^2 t} \qquad \text{Sym}^2\ \frac{t \equiv^2 t'}{t' \equiv^2 t} \qquad \text{Trans}^2\ \frac{t \equiv^2 t' \quad t' \equiv^2 t''}{t' \equiv^2 t}$$

$$\text{Cong}^2\ \frac{t_i \equiv^1 t'_i \quad (1 \le i \le k)}{\mathsf{o}\, t_1 \ldots t_k \equiv^2 \mathsf{o}\, t'_1 \ldots t'_k}\ \mathsf{o} \in \Sigma(k) \qquad \text{Inc}^2\ \frac{t \equiv^1 t'}{t \equiv^2 t'}$$

NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \underset{v}{\overset{u}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma\mathcal{A}} V$$

where $V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_l}$

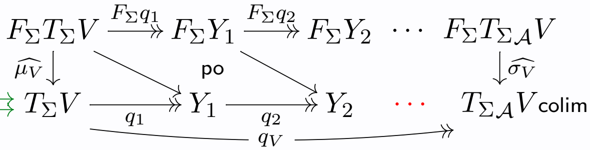
$$T_\Sigma V = \{\, x_1 : n_1, \ldots, x_l : n_l \vdash t \,\}$$

$\forall\, (\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t') \in \mathcal{A}$

$\vdots$

$[\widetilde{\Gamma}, T_\Sigma V]\, \#\, \mathbb{A}^{\#n} \underset{[\![t']\!]}{\overset{[\![t]\!]}{\rightrightarrows}} T_\Sigma V$

$\vdots$



$$\text{Ref}^n \ \frac{}{t \equiv^n t} \qquad \text{Sym}^n \ \frac{t \equiv^n t'}{t' \equiv^n t} \qquad \text{Trans}^n \ \frac{t \equiv^n t' \quad t' \equiv^n t''}{t' \equiv^n t}$$

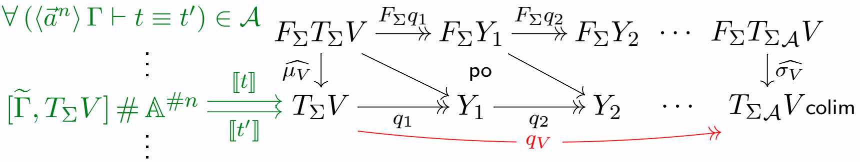
$$\text{Cong}^n \ \frac{t_i \equiv^{n-1} t'_i \quad (1 \leq i \leq k)}{\mathsf{o}\, t_1 \ldots t_k \equiv^n \mathsf{o}\, t'_1 \ldots t'_k}\ \mathsf{o} \in \Sigma(k) \qquad \text{Inc}^n \ \frac{t \equiv^{n-1} t'}{t \equiv^n t'}$$

NTES/L: Completeness

$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v \iff \mathbb{A}^{\#m} \overset{u}{\underset{v}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma\mathcal{A}} V$

where $V = \mathbb{A}^{\#n_1} + \ldots + \mathbb{A}^{\#n_l}$

$T_\Sigma V = \{\, x_1 : n_1, \ldots, x_l : n_l \vdash t \,\}$



$$\text{Ref}^\omega \ \frac{}{t \equiv^\omega t} \qquad \text{Sym}^\omega \ \frac{t \equiv^\omega t'}{t' \equiv^\omega t} \qquad \text{Trans}^\omega \ \frac{t \equiv^\omega t' \quad t' \equiv^\omega t''}{t' \equiv^\omega t}$$

$$\text{Axiom}^\omega \ \frac{(\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t') \in \mathcal{A} \qquad \vec{b}^n \# \{\, \langle \vec{c_x} \rangle\, s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_\Sigma V] \,\}_{x \in |\Gamma|}}{((\vec{a}\,\vec{b}) \cdot t)[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x \equiv^\omega ((\vec{a}\,\vec{b}) \cdot t')[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x}$$

$$\text{Cong}^\omega \ \frac{t_i \equiv^\omega t'_i \quad (1 \le i \le k)}{\mathsf{o}\, t_1 \ldots t_k \equiv^\omega \mathsf{o}\, t'_1 \ldots t'_k} \ \mathsf{o} \in \Sigma(k)$$

NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v$$

$$\Updownarrow$$

$$\mathbb{A}^{\#m} \overset{u}{\underset{v}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma_\mathcal{A}} V$$

$$\Updownarrow$$

$$u \equiv^\omega v$$

$$\text{Ref}^\omega \; \frac{}{t \equiv^\omega t} \qquad \text{Sym}^\omega \; \frac{t \equiv^\omega t'}{t' \equiv^\omega t} \qquad \text{Trans}^\omega \; \frac{t \equiv^\omega t' \quad t' \equiv^\omega t''}{t' \equiv^\omega t}$$

$$\text{Axiom}^\omega \; \frac{\big(\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t'\big) \in \mathcal{A} \qquad \vec{b}^n \,\#\, \{\, \langle \vec{c_x} \rangle\, s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_\Sigma V] \,\}_{x \in |\Gamma|}}{((\vec{a}\,\vec{b}) \cdot t)[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x \equiv^\omega ((\vec{a}\,\vec{b}) \cdot t')[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x}$$

$$\text{Cong}^\omega \; \frac{t_i \equiv^\omega t'_i \quad (1 \le i \le k)}{\mathsf{o}\, t_1 \ldots t_k \equiv^\omega \mathsf{o}\, t'_1 \ldots t'_k}\; \mathsf{o} \in \Sigma(k)$$

NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v$$

$$\Updownarrow$$

$$\mathbb{A}^{\#m} \overset{u}{\underset{v}{\rightrightarrows}} T_\Sigma V \xrightarrow{q_V} T_{\Sigma_\mathcal{A}} V$$

$$\Updownarrow$$

$$u \equiv^\omega v$$

$\Updownarrow$ standard term rewriting

$$u \xleftrightarrow{\text{Axiom}^\omega;(\text{Cong}^\omega)^*} t_1 \longleftrightarrow \cdots \longleftrightarrow t_{n-1} \xleftrightarrow{\text{Axiom}^\omega;(\text{Cong}^\omega)^*} v$$

$$\text{Ref}^\omega\ \frac{}{t \equiv^\omega t} \qquad \text{Sym}^\omega\ \frac{t \equiv^\omega t'}{t' \equiv^\omega t} \qquad \text{Trans}^\omega\ \frac{t \equiv^\omega t' \quad t' \equiv^\omega t''}{t' \equiv^\omega t}$$

$$\text{Axiom}^\omega\ \frac{\big(\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t'\big) \in \mathcal{A} \qquad \vec{b}^n \ \#\ \{\, \langle \vec{c_x} \rangle\, s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_\Sigma V]\,\}_{x \in |\Gamma|}}{((\vec{a}\,\vec{b}) \cdot t)[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x \equiv^\omega ((\vec{a}\,\vec{b}) \cdot t')[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x}$$

$$\text{Cong}^\omega\ \frac{t_i \equiv^\omega t'_i \quad (1 \le i \le k)}{\mathsf{o}\, t_1 \ldots t_k \equiv^\omega \mathsf{o}\, t'_1 \ldots t'_k}\ \mathsf{o} \in \Sigma(k)$$

NTES/L: Completeness

$$\mathcal{A} \models \langle \vec{a}^m \rangle\, x_1 : n_1, \ldots, x_l : n_l \vdash u \equiv v$$

$$\Updownarrow$$

$$\mathbb{A}^{\#m} \xrightarrow{u} T_\Sigma V \xrightarrow{q_V} T_{\Xi}\, V$$

As proofs of $u \equiv^\omega v$ can be turned into proofs in NTEL,

NTEL is complete.

$$u \xleftrightarrow{\text{Axiom}^\omega;(\text{Cong}^\omega)^*} t_1 \longleftrightarrow \cdots \longleftrightarrow t_{n-1} \xleftrightarrow{\text{Axiom}^\omega;(\text{Cong}^\omega)^*} v$$

$$\text{Ref}^\omega \frac{}{t \equiv^\omega t} \qquad \text{Sym}^\omega \frac{t \equiv^\omega t'}{t' \equiv^\omega t} \qquad \text{Trans}^\omega \frac{t \equiv^\omega t' \quad t' \equiv^\omega t''}{t' \equiv^\omega t}$$

$$\text{Axiom}^\omega \frac{\left(\langle \vec{a}^n \rangle\, \Gamma \vdash t \equiv t'\right) \in \mathcal{A} \qquad \vec{b}^n \# \{\, \langle \vec{c_x} \rangle\, s_x \in [\mathbb{A}^{\#\Gamma(x)}, T_\Sigma V]\,\}_{x \in |\Gamma|}}{((\vec{a}\ \vec{b}) \cdot t)[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x \equiv^\omega ((\vec{a}\ \vec{b}) \cdot t')[x \mapsto \langle \vec{c_x} \rangle\, s_x]_x}$$

$$\text{Cong}^\omega \frac{t_i \equiv^\omega t_i' \quad (1 \le i \le k)}{\mathsf{o}\, t_1 \ldots t_k \equiv^\omega \mathsf{o}\, t_1' \ldots t_k'}\ \mathsf{o} \in \Sigma(k)$$

The NTEL is equivalent to the Nominal Equational Logic (Gabbay & Mathijssen 06; Clouston & Pitts 07).

$$
\begin{array}{rcl}
\langle a, b\rangle\, x : 1 \vdash L_a\, x(a) \equiv L_b\, x(b) & \cong & b \,\#\, x \vdash L_a\, x \equiv L_b\, (a\ b)x \\
\langle a\rangle\, x : 0 \vdash L_a\, (A\, x()\, V_a) \equiv x() & \cong & a \,\#\, x \vdash L_a\, (A\, x\, V_a) \equiv x
\end{array}
$$

Discussion

1. Equational logic
   - algebraic theories and first-order equational logic ($\mathcal{S}et$)
   - Nominal Equational Logic (Gabbay & Mathijssen 06; Clouston & Pitts 07) ($\mathcal{N}om$)
   - higher-order equational logic
2. Rewriting system/logic
   - first-order Term Rewriting System ($\mathcal{P}re$)
   - Binding Term Rewriting System (Hamana 03) ($\mathcal{P}re^{\mathbb{I}}$)
   - Nominal Rewriting System (Fernández,Gabbay,Mackie 04) ($\mathrm{Perm}//_{\mathcal{P}(\mathbb{A})}$)
     cf. NTERS ($\mathcal{P}re\mathcal{N}om$)
   - Combinatory Reduction System (Klop 80) ($\mathcal{P}re^{\mathbb{F}}$)
   - Higher-order Rewrite System (Nipkow 91)
   - Maude system (Meseguer *et al.*)