Categorical Equational Systems: Algebraic Models and Equational Reasoning

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This dissertation is submitted for the degree of Doctor of Philosophy

Dedicated to my parents and my wife

Declaration

This dissertation is the result of my own work done under the guidance of my supervisor, and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

This dissertation is not substantially the same as any that I have submitted or will be submitting for a degree or diploma or other qualification at this or any other University.

This dissertation does not exceed the regulation length of 60,000 words, including tables and footnotes.

Summary

We introduce two abstract notions of equational algebraic system, called *Equational System (ES)* and *Term Equational System (TES)*, in order to achieve sufficient expressivity as needed in modern applications in computer science. These generalize the classical concept of (enriched) algebraic theory of Kelly and Power [1993]. We also develop a theory for constructing free algebras for ESs and a theory of equational reasoning for TESs.

In Part I, we introduce the general abstract, yet practical, concept of equational system and develop finitary and transfinitary conditions under which we give an explicit construction of free algebras for ESs. This free construction extends the well-known construction of free algebras for ω-cocontinuous endofunctors to an equational setting, capturing the intuition that free algebras consist of freely constructed terms quotiented by given equations and congruence rules. We further show the monadicity and cocompleteness of categories of algebras for ESs under the finitary and transfinitary conditions. To illustrate the expressivity of equational systems, we exhibit various examples including two modern applications, the Σ-monoids of Fiore et al. [1999] and the π-algebras of Stark [2005].

In Part II, we introduce the more concrete notion of term equational system, which is obtained by specializing the concept of equational system, but remains more general than that of enriched algebraic theory. We first develop a sound logical deduction system, called *Term Equational Logic (TEL)*, for equational reasoning about algebras of TESs. Then, to pursue a complete logic, we give an *internal completeness* result, from which together with the explicit construction of free algebras one can typically synthesize sound and complete rewriting-style equational logics. To exemplify this scenario, we give two applications: multi-sorted algebraic theories and nominal equational theories of Clouston and Pitts [2007] and of Gabbay and Mathijssen [2007].

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Chapter 1

Introduction

Algebraic structures satisfying equational constraints commonly arise in theoretical computer science. For instance, in the algebraic specification of abstract data types [Goguen et al. 1978], the algebraic treatment of computational effects [Plotkin and Power 2003, 2004], and the algebraic modeling of the π-calculus [Stark 2005, 2008]. For such algebras, one is often interested in properties such as the existence of free algebras and equational reasoning about them.

Algebraic theories (see *e.g.* [Wraith 1975]) are a classical framework for defining equational algebras that enjoy the aforementioned properties: free algebras for every algebraic theory exist and there is a sound and complete equational logic for reasoning about them. These algebraic theories are restricted to a set-theoretic notion of algebra. To overcome this the notion of *enriched algebraic theory* was introduced by Kelly and Power [1993]. While this generalizes the classical notion of algebraic theory into enriched settings and admits algebras based on other categories as well as the category **Set** of sets and functions, it lacks an equational logic for reasoning about algebraic structures. Furthermore, enriched algebraic theories are not expressive enough to directly accommodate several recent applications: Σ-monoids [Fiore et al. 1999], π-algebras [Stark 2005] and nominal equational theories [Clouston and Pitts 2007].

This thesis overcomes these problems. Firstly, we propose an abstract notion of algebraic equational system, called *Equational System (ES)*. ESs, being abstract and general, can accommodate the above recent applications as well as enriched algebraic theories. We develop sufficient conditions for the existence of free algebras for ESs and an explicit construction of them under these conditions. Secondly, we propose a more concrete notion of algebraic equational system, called *Term Equational System (TES)*, for which we study a general theory of equational reasoning. TESs are obtained by specializing ESs, but remain more general than enriched algebraic theories in expressivity. Thus, they enjoy all properties of ESs. We first develop a general sound equational logic for TESs. Then, to pursue a complete logic, we give an *internal completeness* result, from which together with the explicit construction of free algebras one can typically synthesize sound and complete equational logics.

1.1 Background

To motivate our work, we briefly review the two classical frameworks: algebraic theories and enriched algebraic theories.

1.1.1 Algebraic theories

An algebraic theory consists of a *signature* specifying the structure of its algebras and a set of equations, called *axioms*, yielding equational constraints that the algebras should satisfy. A signature is given by a set of operators O with an arity function $|-| : \mathsf{O} \to \mathbb{N}$ assigning an arity to each operator. An equation of arity n for a signature is given by a pair of terms built up from n distinct variables and the operators of the signature.

A typical example is the theory of groups $\mathbb{G} = (\Sigma_{\mathbb{G}}, E_{\mathbb{G}})$. The signature $\Sigma_{\mathbb{G}}$ consists of three operators: the identity e of arity 0, the inverse i of arity 1, and the multiplication m of arity 2. This signature specifies the algebraic structure of groups: an algebra for the signature $\Sigma_{\mathbb{G}}$ is a carrier set X equipped with interpretation functions $[\![\mathsf{e}]\!] : 1 \to X$, $[\![\mathsf{i}]\!] : X \to X$, $[\![\mathsf{m}]\!] : X^2 \to X$ of the three operators. The set $E_{\mathbb{G}}$ consists of equations expressing the usual group axioms. For instance, the equation for the associativity of multiplication has arity 3 and is given by the following two terms with variables x, y, z:

$$\{\, x, y, z \,\} \;\vdash\; \mathsf{m}(\mathsf{m}(x, y), z) \;\equiv\; \mathsf{m}(x, \mathsf{m}(y, z)) \ .$$

This equation induces the following constraint on algebras $(X, [\![\mathsf{e}]\!], [\![\mathsf{i}]\!], [\![\mathsf{m}]\!])$ for $\Sigma_{\mathbb{G}}$:

$$\text{for all } x, y, z \in X, \quad [\![\mathsf{m}]\!]([\![\mathsf{m}]\!](x, y), z) \;=\; [\![\mathsf{m}]\!](x, [\![\mathsf{m}]\!](y, z)) \ .$$

Algebras for the theory of groups $\mathbb{G}$ (also called $\mathbb{G}$-algebras) are algebras for the signature $\Sigma_{\mathbb{G}}$ satisfying the equational constraints induced from the equations of $E_{\mathbb{G}}$. These indeed define the usual notion of group.

The following are well-known properties of algebraic theories $\mathbb{T}$:

- free $\mathbb{T}$-algebras on sets exist and their construction is explicitly described;
- the category of $\mathbb{T}$-algebras is monadic over the category **Set**, in the sense that the category of $\mathbb{T}$-algebras is isomorphic to the category of Eilenberg-Moore algebras for the monad induced by free $\mathbb{T}$-algebras; and
- the category of $\mathbb{T}$-algebras is complete and cocomplete.

Traditional computer science applications of algebraic theories include the initial algebra approach to the semantics of computational languages and the specification of abstract data types pioneered by the ADJ group [Goguen et al. 1978], and the abstract description of powerdomain constructions as free algebras of non-determinism advocated by Plotkin [Hennessy and Plotkin 1979, Plotkin 1983] (see also [Abramsky and Jung 1994]).

There is a sound and complete equational logic for algebraic theories, called *equational logic* (see *e.g.* [Birkhoff 1935, Goguen and Meseguer 1985]). The logic is sound in the sense that any equation derived by means of the logic for an algebraic theory is *valid* (*i.e.*, satisfied by all algebras of the theory); and it is complete in the sense that every valid equation is derivable. The logic consists of the rule Axiom stating that the axioms of a theory are valid, and the rule Subst stating that the equality is preserved under substitution, together with the equivalence relation rules Ref, Sym and Trans (see Example 7.1.1 in Section 7.1 for the definition of the rules). For instance, we can deduce the equation $\emptyset \vdash \mathsf{e} \equiv \mathsf{i}(\mathsf{e})$ as follows:

$$
\mathsf{Trans}\,\dfrac{\mathsf{Sym}\,\dfrac{\mathsf{Subst}\,\dfrac{\mathsf{Axiom}\,\dfrac{}{\{x\} \vdash \mathsf{m}(x,\mathsf{i}(x)) \equiv \mathsf{e}} \quad x \mapsto \mathsf{Ref}\,\dfrac{}{\emptyset \vdash \mathsf{e} \equiv \mathsf{e}}}{\emptyset \vdash \mathsf{m}(\mathsf{e},\mathsf{i}(\mathsf{e})) \equiv \mathsf{e}}}{\emptyset \vdash \mathsf{e} \equiv \mathsf{m}(\mathsf{e},\mathsf{i}(\mathsf{e}))} \qquad \dfrac{\vdots}{\emptyset \vdash \mathsf{m}(\mathsf{e},\mathsf{i}(\mathsf{e})) \equiv \mathsf{i}(\mathsf{e})}}{\emptyset \vdash \mathsf{e} \equiv \mathsf{i}(\mathsf{e})}
$$

Algebraic theories also support equational reasoning by rewriting, called *Term Rewriting* (see *e.g.* [Baader and Nipkow 1999]), which is better adapted to mechanization. For instance, the above equation $\emptyset \vdash \mathsf{e} \equiv \mathsf{i}(\mathsf{e})$ can be derived from the following two rewriting steps:

$$\mathsf{e} \longrightarrow \mathsf{m}(\mathsf{e},\mathsf{i}(\mathsf{e})) \longrightarrow \mathsf{i}(\mathsf{e})$$

where the first step is obtained from the axiom $\{x\} \vdash \mathsf{m}(x,\mathsf{i}(x)) \equiv \mathsf{e}$ by swapping the two terms and substituting the variable x with the term e; and the second step from the axiom $\{x\} \vdash \mathsf{m}(\mathsf{e},x) \equiv x$ by substituting the variable x with the term $\mathsf{i}(\mathsf{e})$.

Relationship with Lawvere theories and finitary monads. The notion of algebraic theory has a strong connection to more abstract concepts of Lawvere theory [Lawvere 1963] and finitary monad.

A Lawvere theory $\mathscr{L}$ is a category with a countable set $\{C^0, C^1, \ldots, C^n, \ldots\}$ of distinct objects such that each object C^n is the n-th power of the object C^1 (*i.e.*, $C^n = C^1 \times \ldots \times C^1$ (n times)). A model of the theory $\mathscr{L}$ is a product preserving functor from $\mathscr{L}$ to **Set**. A homomorphism of $\mathscr{L}$-models is a natural transformation.

It is well known that the three concepts of algebraic theory, Lawvere theory and finitary monad on **Set** (*i.e.*, monads preserving filtered colimits) are equivalent in the following sense (see *e.g.* [Borceux 1994]):

- for every algebraic theory $\mathbb{T}$, there exists a Lawvere theory $\mathscr{L}$ such that the category of $\mathbb{T}$-algebras is isomorphic to that of $\mathscr{L}$-models;
- for every Lawvere theory $\mathscr{L}$, there exists a finitary monad $\mathbf{T}$ on **Set** such that the category of $\mathscr{L}$-models is isomorphic to that of Eilenberg-Moore algebras for $\mathbf{T}$; and

- for every finitary monad $\mathbf{T}$ on **Set**, there exists an algebraic theory $\mathbb{T}$ such that the category of Eilenberg-Moore algebras for $\mathbf{T}$ is isomorphic to that of $\mathbb{T}$-algebras.

1.1.2 Enriched algebraic theories

Kelly and Power [1993] introduced the notion of enriched algebraic theory, which directly generalizes that of algebraic theory into an enriched categorical setting (see [Kelly 1982]).

First, the base category is generalized from the category **Set** to any enriched category $\mathscr{C}$ with suitable structure. Technically, the base category should be a locally finitely presentable category enriched over a symmetric monoidal closed category that is locally finitely presentable as a closed category. For the purpose of this introduction, we simply consider symmetric monoidal closed (SMC) categories with the required structure (*i.e.*, locally finite presentability as a closed category) as base categories for enriched algebraic theories.

Given such a base category $\mathscr{C}$ with a SMC structure $(\otimes, I, [-,=])$, the notion of arity is generalized from a natural number to a finitely presentable object in the base category $\mathscr{C}$ and the new notion of *coarity* is given as an object in $\mathscr{C}$. The notion of operator is accordingly generalized to have an arity and a coarity. Indeed, operators of arity n in algebraic theories become operators of arity $\{1,\ldots,n\}$ and coarity $\{1\}$ in enriched algebraic theories based on the category **Set**.

A signature Σ for an enriched algebraic theory on the base category $\mathscr{C}$ is given as a set of operators with arity and coarity in $\mathscr{C}$. An algebra for the signature is given by a carrier object X in the category $\mathscr{C}$, equipped with an interpretation map

$$[\![\mathsf{o}]\!] : [A, X] \otimes C \to X$$

for each operator o of arity A and coarity C. The category Σ-**Alg** of algebras for the signature Σ is monadic over the base category $\mathscr{C}$ along the forgetful functor $\Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ sending an algebra to its carrier object. Thus a monad $\mathbf{T}_\Sigma$ on $\mathscr{C}$, called *term monad*, is induced. It further carries an internal functor structure, or equivalently, a strong monad structure with a strength

$$\mathsf{st}_{X,Y} : X \otimes \mathbf{T}_\Sigma Y \to \mathbf{T}_\Sigma(X \otimes Y)\ .$$

The term monad $\mathbf{T}_\Sigma$ admits *interpretation maps* $\mathbf{T}_\Sigma V \to [[V,X],X]$ for all objects V and Σ-algebras $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma})$, given as the transpose of the composite

$$[V,X] \otimes \mathbf{T}_\Sigma V \xrightarrow{\mathsf{st}_{[V,X],V}} \mathbf{T}_\Sigma([V,X]\otimes V) \xrightarrow{\mathbf{T}_\Sigma(\epsilon)} \mathbf{T}_\Sigma X \xrightarrow{\overline{[\![\cdot]\!]}} X$$

where $(X, \overline{[\![\cdot]\!]} : \mathbf{T}_\Sigma X \to X)$ is the Eilenberg-Moore algebra for $\mathbf{T}_\Sigma$ corresponding to the Σ-algebra $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma})$.

A term of arity A and coarity C for a signature Σ is defined as a morphism $C \to \mathbf{T}_\Sigma A$ and an equation as a pair of terms of the same arity and coarity. A Σ-algebra $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma})$

is then said to satisfy an equation $t_1 \equiv t_2 : C \to \mathbf{T}_\Sigma A$ whenever the interpretation map $\mathbf{T}_\Sigma A \to [[A, X], X]$ given as above coequalizes the maps t_1 and t_2.

An enriched algebraic theory is given by a pair consisting of a signature and a set of equations; and its algebras are algebras for the signature satisfying the equations. Similarly for algebraic theories, each enriched algebraic theory $\mathbb{T}$ on a base category $\mathscr{C}$ satisfies the following properties:

- the category of $\mathbb{T}$-algebras is monadic over $\mathscr{C}$; and
- the category of $\mathbb{T}$-algebras is complete and cocomplete.

Although monadicity implies the existence of free algebras on objects in $\mathscr{C}$, it does not provide an explicit construction that captures the following usual intuition: free algebras for a theory consist of terms built up from variables and the operators of its signature, quotiented by the axioms of the theory. Also, no equational logic generally applicable to enriched algebraic theories has been developed.

As for algebraic theories, the equivalence between enriched algebraic theories, enriched Lawvere theories and finitary enriched monads holds (see [Kelly and Power 1993] and [Power 1999]).

The algebraic treatment of computational effects [Plotkin and Power 2003, 2004] is an application of enriched algebraic theories. However, as already mentioned, some recent applications need our more general framework.

1.2 Approach and contributions

Motivated by the limitations of enriched algebraic theories, we aim to achieve the following goals in the development of our new notion of algebraic equational system:

- increase expressiveness to accommodate recent applications;
- develop a simple and explicit construction of free algebras that directly reflects the usual intuition; and
- provide a general sound and complete equational logic for this extended notion of algebraic equational systems.

For these purposes, we propose two frameworks, called Equational Systems (ESs) and Term Equational Systems (TESs). TESs generalize enriched algebraic theories and accommodate nominal equational theories [Clouston and Pitts 2007] as instances (see Section 8.2). ESs further generalize TESs and accommodate π-algebras [Stark 2005] and Σ-monoids [Fiore et al. 1999] (see Sections 5.1 and 5.2). On the other hand, we develop an explicit construction of free algebras for ESs and an equational logic for TESs.

1.2.1 Equational systems

For the purpose of expressiveness, we introduce abstract notions of signature and equation, leading to a new concept of equational system.

Signatures as endofunctors. To motivate our general notion of signature, we quickly discuss the difficulties in representing the concepts of Σ-monoid and of π-algebra as enriched algebraic theories.

A Σ-monoid consists of a carrier object X in a monoidal category $(\mathscr{C}, I, \otimes)$ with certain algebra structure maps, one of which is the monoid multiplication $[\![\mathsf{m}]\!] : X \otimes X \to X$. However, the map is in general hard or impossible to be decomposed into a family of maps of the form $[A, X] \otimes C \to X$.

On the other hand, π-algebras highlight another kind of difficulty: an enriched algebraic theory on a category $\mathscr{C}$ has to be based on a single enrichment (or SMC) structure of $\mathscr{C}$, while the theory of π-algebras is based on two enrichments together. The base category for π-algebras is the functor category $\mathbf{Set}^{\mathbb{I}}$ for $\mathbb{I}$ the (essentially small) category of finite sets and injections. The category $\mathbf{Set}^{\mathbb{I}}$ carries the cartesian closed structure $\big(1, \times, (=)^{(-)}\big)$ and another symmetric monoidal closed structure $\big(1, \otimes, (-) \multimap (=)\big)$. π-algebras consist of a carrier object X in $\mathbf{Set}^{\mathbb{I}}$ together with interpretations of the operators of the finite π-calculus, satisfying relevant equations. Among those operators, the operator new has an interpretation map of the form $(A \multimap X) \otimes C \to X$; while the others have maps of the form $X^A \times C \to X$.

To cope with these problems, we abstractly define arities of operators as endofunctors F on the base category $\mathscr{C}$, called *functorial arities*. An interpretation of an operator with functorial arity F is given as an F*-algebra* (X, s) consisting of a carrier object $X \in \mathscr{C}$ together with an algebra structure map $s : FX \to X$. Note that no enrichment or monoidal structure is required on the base category $\mathscr{C}$.

It is clear that the interpretations of the operators discussed so far form algebras for appropriate endofunctors. For instance, the operator m of Σ-monoid on a category $\mathscr{C}$ has as functorial arity the endofunctor $(-) \otimes (-)$ on $\mathscr{C}$. An operator of arity A and coarity C in an enriched algebraic theory on a category $\mathscr{C}$ has as functorial arity the endofunctor $[A, -] \otimes C$ on $\mathscr{C}$.

A functorial signature is naturally defined as a set of operators with functorial arities. However, when the base category has coproducts, a functorial signature Σ can be simply represented as an operator of functorial arity $\coprod_{\mathsf{o} \in \Sigma} |\mathsf{o}|(-)$, where $|\mathsf{o}|$ denotes the functorial arity of an operator o, because their interpretations are equivalent. As we will mainly consider cocomplete base categories, we simply define a *functorial signature* as a single functorial arity, *i.e.*, an endofunctor on a base category.

Equations as parallel pairs of functors. We propose a notion of equation which is at the same level of abstraction as functorial signature. To motivate it, consider the associativity axiom for the operator m of Σ-monoids, given as the commutativity of the following diagram:

$$\begin{tikzcd}
(X \otimes X) \otimes X \arrow[r, "\alpha_{X,X,X}"] \arrow[d, "\llbracket \mathsf{m} \rrbracket \otimes X"] & X \otimes (X \otimes X) \arrow[r, "X \otimes \llbracket \mathsf{m} \rrbracket"] & X \otimes X \arrow[d, "\llbracket \mathsf{m} \rrbracket"] \\
X \otimes X \arrow[rr, "\llbracket \mathsf{m} \rrbracket"] & & X
\end{tikzcd}$$

We can simply view this as follows. Given an algebra $\llbracket \mathsf{m} \rrbracket : X \otimes X \to X$ for the functorial signature $(-) \otimes (-)$, the associativity axiom induces a pair of algebras

$$(X \otimes X) \otimes X \xrightarrow{\alpha_{X,X,X}} X \otimes (X \otimes X) \xrightarrow{X \otimes \llbracket \mathsf{m} \rrbracket} X \otimes X \xrightarrow{\llbracket \mathsf{m} \rrbracket} X \ ,$$

$$(X \otimes X) \otimes X \xrightarrow{\llbracket \mathsf{m} \rrbracket \otimes X} X \otimes X \xrightarrow{\llbracket \mathsf{m} \rrbracket} X$$

for the endofunctor $((-) \otimes (-)) \otimes (-)$ and requires that the two algebras coincide. In this view, an equation for a functorial signature Σ on $\mathscr{C}$ is given by an endofunctor Γ on $\mathscr{C}$—seen as its arity—and a pair of functors $L, R : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$. Furthermore, the functors L, R should preserve carrier objects, meaning that $U_\Gamma\, L = U_\Sigma$ and $U_\Gamma\, R = U_\Sigma$ for the forgetful functors $U_\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ and $U_\Gamma : \Gamma\text{-}\mathbf{Alg} \to \mathscr{C}$. We call this *a functorial equation*. It is easily seen that the notion of functorial equation is general enough to express the equations discussed so far.

To sum up, an *equational system*

$$\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$$

is given by a functorial equation $\Gamma \vdash L \equiv R$ for a functorial signature Σ on a category $\mathscr{C}$. As mentioned above, although we defined an equational system to consist of a single functorial equation, there is not much loss of generality because one can encode a set of functorial equations $\{\mathscr{C} : \Sigma \rhd \Gamma_i \vdash L_i \equiv R_i\}_{i \in I}$ into the single one $(\mathscr{C} : \Sigma \rhd \coprod_{i \in I} \Gamma_i \vdash [L_i]_{i \in I} \equiv [R_i]_{i \in I})$ whenever $\mathscr{C}$ has coproducts. $\mathbb{S}$-algebras are Σ-algebras $(X, s : \Sigma X \to X)$ such that $L(X, s) = R(X, s) : \Gamma X \to X$ and the category $\mathbb{S}\text{-}\mathbf{Alg}$ of $\mathbb{S}$-algebras is the full subcategory of $\Sigma\text{-}\mathbf{Alg}$ consisting of $\mathbb{S}$-algebras:

$$\begin{tikzcd}
\mathbb{S}\text{-}\mathbf{Alg} \arrow[r, hook, "J_\mathbb{S}"] \arrow[dr, "U_\mathbb{S}"'] & \Sigma\text{-}\mathbf{Alg} \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] \arrow[d, "U_\Sigma"] & \Gamma\text{-}\mathbf{Alg} \arrow[dl, "U_\Gamma"] \\
& \mathscr{C} &
\end{tikzcd}$$

Free constructions for equational systems. The main theory for equational systems is the explicit construction of free algebras. For an ES $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$, since the forgetful functor $U_\mathbb{S}$ decomposes as the composite $U_\Sigma\, J_\mathbb{S}$ as shown in the above diagram, the construction of free $\mathbb{S}$-algebras on objects in $\mathscr{C}$ can be considered in two stages:

(i) the construction of free Σ-algebras on objects in $\mathscr{C}$, and (*ii*) the construction of free $\mathbb{S}$-algebras over Σ-algebras.

The construction (*i*) is well established in the literature (see *e.g.* [Adámek 1974]). For a cocomplete category $\mathscr{C}$ and an endofunctor Σ on $\mathscr{C}$ preserving colimits of ω-chains, a free Σ-algebra on an object X in $\mathscr{C}$ has as carrier the colimit $\mathbf{T}_\Sigma X$ of the ω-chain $\{\, e_n : X_n \to X_{n+1} \,\}_{n \geq 0}$ defined by setting $X_0 = 0$ and $X_{n+1} = X + \Sigma X_n$; and $e_0 = \,! : 0 \to X_1$ and $e_{n+1} = \mathrm{id}_X + \Sigma e_n : X + \Sigma X_n \to X + \Sigma X_{n+1}$, where 0 denotes an initial object and ! the unique map.

$$0 \xrightarrow{\;!\;} X + \Sigma 0 \xrightarrow{X+\Sigma(!)} X + \Sigma(X + \Sigma 0) \longrightarrow \cdots\cdots \longrightarrow \mathbf{T}_\Sigma X \tag{1.1}$$

Intuitively this captures the construction of freely generated terms with operators from Σ and variables from X.

Our contribution is the development of the construction (*ii*). We give sufficient conditions for the existence of free $\mathbb{S}$-algebras over Σ-algebras and provide an explicit construction of the free algebras under the conditions. Among those conditions, the following are the most important.

- *(κ-finitary)* The category $\mathscr{C}$ is cocomplete and the endofunctors Σ and Γ preserve colimits of κ-chains for some infinite limit ordinal κ.

- *(κ-inductive)* The ES $\mathbb{S}$ is κ-finitary and additionally the endofunctors Σ and Γ preserve epimorphisms.

We can construct free algebras in $\kappa \times \kappa$ steps for κ-finitary ESs, but in κ steps for κ-inductive ESs. For instance, for an ω-inductive equational system

$$\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R),$$

the free $\mathbb{S}$-algebra $(\widetilde{X}, \widetilde{s})$ over a Σ-algebra (X, s) is constructed as follows:

$$\begin{tikzcd}
 & \Sigma X \arrow[r, two heads, "\Sigma q_0"] \arrow[d, "s"'] \arrow[dr, "s_0"] & \Sigma X_1 \arrow[r, two heads, "\Sigma q_1"] \arrow[d, phantom, "\mathsf{po}"] \arrow[dr, "s_1"] & \Sigma X_2 \arrow[r, two heads, "\Sigma q_2"] \arrow[d, phantom, "\mathsf{po}"] \arrow[dr, "s_2"] & \Sigma X_3 & \cdots\cdots & \Sigma \widetilde{X} \arrow[d, dashed, "\exists! \widetilde{s}"] \\
\Gamma X \arrow[r, shift left, "L(X,s)"] \arrow[r, shift right, "R(X,s)"'] & X \arrow[r, two heads, "q_0", "\mathsf{coeq}"'] & X_1 \arrow[r, two heads, "q_1"] & X_2 \arrow[r, two heads, "q_2"] & X_3 & \cdots\cdots & \underset{\mathsf{colim}}{\widetilde{X}}
\end{tikzcd} \tag{1.2}$$

where the map q_0 is a coequalizer of $L(X, s), R(X, s)$, the map s_0 is the composite $q_0 \circ s$, and the cospan (s_{n+1}, q_{n+1}) is a pushout of $(\Sigma q_n, s_n)$ for all $n \geq 0$; and where $\widetilde{X}$ is a colimit of the ω-chain $\{\, q_n \,\}_{n \geq 0}$ and, as Σ preserves the colimit, $\widetilde{s}$ is the unique mediating map from the colimit $\Sigma \widetilde{X}$ of the chain $\{\, \Sigma q_n \,\}_{n \geq 0}$. Intuitively the map q_0 captures the construction of quotienting X by the equation $L = R$, and the maps $\{\, q_n \,\}_{n \geq 1}$ capture that of iteratively quotienting it by congruence rules for the operators of Σ.

For κ-finitary/κ-inductive equational systems $\mathbb{S}$, the following properties hold:

- the category $\mathbb{S}$-**Alg** of $\mathbb{S}$-algebras is monadic over $\mathscr{C}$; and

- the category $\mathbb{S}$-**Alg** of $\mathbb{S}$-algebras is cocomplete.

As every enriched algebraic theory induces an equivalent ω-finitary equational system (see Section 2.5), the properties of enriched algebraic theories discussed in Section 1.1.2 follow as corollary from the above properties of ω-finitary equational systems.

Advantages of equational systems. Besides their expressivity, equational systems have further benefits over enriched algebraic theories.

- The locally finite presentability is not required for base categories. Examples of cocomplete but not locally finitely presentable categories include the category of topological spaces, the category of directed-complete posets, and the category of complete semilattices.

- The concept of equational system is straightforwardly dualizable: an equational cosystem on a category is simply an equational system on the opposite category. Thus, for instance, comonoids in a monoidal category arise as coalgebras for an equational cosystem. (See Sections 2.4 and 2.5.)

On the other hand, the price paid for all this generality is that the important connection between enriched algebraic theories and enriched Lawvere theories [Power 1999] is lost for equational systems.

1.2.2 Term equational systems

The notion of equational system is so general that it is hard to develop an equational logic for it. Thus we give a more concrete notion of algebraic equational system, called Term Equational System (TES), and study equational reasoning for TESs (see Section 6.5 for the relation of TES to ES). The notion of TES is still more general than that of enriched algebraic theories, in the sense that every enriched algebraic theory can be expressed as a TES.

Term equational system. The notion of TES generalizes that of enriched algebraic theory as follows.

- *(Base category)* By removing the locally finite presentability condition from enriched algebraic theories, we define a base category for a TES to be a tensored and cotensored category enriched over a symmetric monoidal closed category. Indeed, this notion is further generalized to a bi-closed action of a monoidal category (see Section 6.1 for definition). Typical examples are symmetric monoidal closed (SMC) categories, as they are bi-closed monoidal actions of themselves. For the purpose of this introduction, we simply consider SMC categories as base categories for TESs. However, note that the general notion of base category as a bi-closed action of a monoidal category is indispensable to accommodate multi-sorted theories (see Section 8.1).

- *(Signature)* Recall that every signature Σ for an enriched algebraic theory induces a strong monad $\mathbf{T}_\Sigma$ on its base category, and that the category of algebras for the signature Σ is isomorphic to the category of Eilenberg-Moore algebras for the monad $\mathbf{T}_\Sigma$. Thus, we generally define a signature for a TES as a strong monad on its base category and an algebra for the signature as an Eilenberg-Moore algebra for the monad.

- *(Equation)* An equation for a TES is defined essentially in the same way as that for an enriched algebraic theory. An equation of arity A and coarity C for a strong monad $(\mathbf{T}, \mathsf{st})$ is given by a pair of morphisms $t_1, t_2 : C \to \mathbf{T}A$ and an Eilenberg-Moore algebra $(X, s : \mathbf{T}X \to X)$ satisfies it if the following diagram commutes:

$$[A,X]\otimes C \underset{[A,X]\otimes t_2}{\overset{[A,X]\otimes t_1}{\rightrightarrows}} [A,X]\otimes \mathbf{T}A \xrightarrow{\mathsf{st}_{[A,X],A}} \mathbf{T}([A,X]\otimes A) \xrightarrow{\mathbf{T}(\epsilon)} \mathbf{T}X \xrightarrow{s} X\ .$$

In summary, a TES

$$\mathbb{S} = ((\mathscr{C}, \otimes, I, [-,=]), (\mathbf{T}, \mathsf{st}), E)$$

is given by a SMC category $(\mathscr{C}, \otimes, I, [-,=])$, a strong monad $\mathbf{T}$ on $\mathscr{C}$ with strength st, and a set E of equations. An $\mathbb{S}$-algebra is given by an Eilenberg-Moore algebra for the monad $\mathbf{T}$ satisfying the equations in E.

Equational reasoning by deduction. We present a sound deduction system, called Term Equational Logic (TEL), for deriving valid equations for TESs. The logic consists of

- the rules Ref, Sym, Trans of equivalence relations;
- the rule Axiom stating that the axioms of a given theory are valid;
- the rule Subst stating that substitution is a congruence;
- the rule Ext stating that an operation of context extension is a congruence; and
- the rule Local expressing the local character of entailment.

The formal definition of TEL is given in Section 7.1.1.

Then we prove the soundness of TEL: every equation derived from the axioms of a TES $\mathbb{S}$ by means of TEL is satisfied by all $\mathbb{S}$-algebras. However, we do not have a general completeness result—the converse of soundness—for TEL.

Equational reasoning by rewriting. We pursue a complete rewriting logic. First of all, we establish a result that simplifies the validity condition of an equation as follows:

> For a TES $\mathbb{S}$ that admits free algebras, an equation $u \equiv v : C \to \mathbf{T}A$ is satisfied by all $\mathbb{S}$-algebras if and only if it is satisfied by the free $\mathbb{S}$-algebra on the object A.

We call this theorem the *internal completeness* of TESs.

As every TES induces an equivalent ES, from the theory developed for ESs we obtain sufficient conditions for the existence of free algebras for TESs and further an explicit construction of them. For instance, a TES $\mathbb{S} = (\mathscr{C}, \mathbf{T}, E)$ is said to be *κ-inductive* for an infinite limit ordinal κ if

- the base category $\mathscr{C}$ is cocomplete,
- the monad $\mathbf{T}$ preserves epimorphisms and colimits of κ-chains, and
- every equation $C \to \mathbf{T}A$ in E has *projective* and *κ-compact* arity, meaning that the endofunctor $[A, -]$ on $\mathscr{C}$ preserves epimorphisms and colimits of κ-chains.

Indeed, κ-inductive TESs induce κ-inductive ESs and thus free algebras for κ-inductive TESs are constructed in κ steps. If the signature of an ω-inductive TES arises as a free monad $\mathbf{T}_\Sigma$ on an endofunctor Σ preserving epimorphisms and colimits of ω-chains, then free algebras for the TES are inductively constructed as in (1.1) followed by (1.2).

Although we do not yet have a general complete logic for TESs, as we shall see through the applications of Chapter 8, for concrete instances of ω-inductive TESs one may directly extract a sound and complete logic from the inductive construction of free algebras using the internal completeness result. Furthermore, as the construction (1.2) quotients the carrier object by axioms and congruence rules, the extracted complete logic only consists of an axiom rule and a congruence rule together with equivalence relation rules. An advantage of having only those rules is that it supports equational reasoning by rewriting, which is well suited for mechanization.

One may either establish the completeness of the TEL associated to such TESs by turning each rewrite step $u \to v$ of the extracted complete logic into a proof of the equation $u \equiv v$ in TEL, or get insight into how to extend it to make it complete.

Applications. We advocate the following general methodology for developing term equational systems and logics.

1. Select a cocomplete SMC category $\mathscr{C}$ as a base category and consider within it a notion of signature such that every signature Σ gives rise to a strong monad $\mathbf{T}_\Sigma$ on $\mathscr{C}$ preserving epimorphisms and colimits of ω-chains.
2. Select a class of arities A and coarities C such that the arities A are projective and ω-compact, and give a syntactic description of morphisms $C \to \mathbf{T}_\Sigma A$. This yields a

syntactic notion of equational theory with an associated model theory arising from that of the underlying term equational system.

3. Synthesize a deduction system for equational reasoning on syntactic terms with rules arising as syntactic counterparts of the rules from the term equational logic associated to the underlying term equational system. By construction, soundness will be guaranteed.

4. In view of the internal completeness result, analyze the inductive construction of free algebras to synthesize a complete equational logic by rewriting. This complete logic may be used to show the completeness of the above equational logic arising from TEL.

Existing equational theories that arise as TESs and for which we can develop equational logics following the above methodology include

- algebraic theories (see *e.g.* [Wraith 1975]),
- nominal equational theories [Clouston and Pitts 2007, Gabbay and Mathijssen 2007],
- binding term equational theories [Hamana 2003],
- second-order algebraic theories [Fiore 2008]; and
- multi-sorted versions of the above.

This methodology in the cartesian closed category **Set** with the term monad $\mathbf{T}_\Sigma$ induced from a signature Σ for algebraic theories, and equations of arity $\{1, \ldots, n\}$ and coarity $\{1\}$ for $n \in \mathbb{N}$ leads to the equational logic and the term rewriting system for algebraic theories. (See the running example of Chapters 6 and 7.)

The methodology for multi-sorted algebraic theories is more interesting. Goguen and Meseguer [1985] pointed out that a naive generalization of equational reasoning by rewriting for single-sorted algebraic theories is not sound for multi-sorted algebraic theories, and proposed a sound and complete equational logic by deduction for multi-sorted algebraic theories. Indeed, the equational logic of Goguen and Meseguer arises as TEL for multi-sorted algebraic theories. Furthermore, the complete logic extracted from the construction of free algebras fixes the naive equational reasoning by rewriting and gives a sound and complete rewriting-style logic. (See Section 8.1.)

The methodology applied within the category **Nom** of nominal sets (which is equivalent to the Schanuel topos) gives rise to equational logics for nominal equational theories. The equational logic arising from the term equational logic is equivalent to the nominal equational logics of Clouston and Pitts [2007] and of Gabbay and Mathijssen [2007]. Interestingly, the notion of nominal rewriting [Fernández et al. 2004, Fernández and Gabbay 2007]—seen as an equational logic—is sound, but not complete with respect to the model theory of nominal equational theories. However, the rewriting-style logic extracted from the construction of free algebras gives rise to a sound and complete rewriting system for nominal equational theories. (See Section 8.2.)

A similar development can be carried out in the category $\mathbf{Set}^{\mathbb{I}}$, for $\mathbb{I}$ the category of finite sets and injections, and this leads to the binding term equational logic and rewriting system of Hamana [2003]. Although the original formulations of nominal equational logic and of binding term equational logic look quite different, one can easily see that they are closely related by viewing them as TESs based on $\mathbf{Nom}$ and its supercategory $\mathbf{Set}^{\mathbb{I}}$. More specifically, the equational logics synthesized by the above methodology applied in the categories $\mathbf{Nom}$ and $\mathbf{Set}^{\mathbb{I}}$ are identical except that the logic based on $\mathbf{Nom}$ has one more inference rule reflecting the difference between $\mathbf{Nom}$ and $\mathbf{Set}^{\mathbb{I}}$. (See Section 8.2.8 for discussion.)

Application of the methodology in the context of second-order abstract syntax as developed in [Fiore 2008] to synthesize an equational logic for second-order algebraic theories is briefly discussed in Section 9.3. This will be further investigated with Fiore and published elsewhere.

1.3 Synopsis

This thesis is split into two parts. We develop the concept of equational system and its associated theory in the first part and the concept of term equational system and its associated theory in the second part.

Part I. Equational Systems and Free Constructions.

Chapter 2: Equational systems. We motivate and define the notions of functorial signature and equation that lead to the concept of equational system, and its model theory. We also introduce variants of equational system: monadic equational system and equational cosystem. The chapter concludes with various examples of equational system illustrating its expressivity.

Chapter 3: Theory of inductive equational systems. We present a simple condition, called inductiveness, under which an inductive construction of free algebras for equational systems is given. Under this condition, categories of algebras for equational systems are also shown to be cocomplete and monadic over their base categories.

Chapter 4: General theory of equational systems. We present more general conditions under which transfinite inductive constructions of free algebras for equational systems are given. Under these conditions, categories of algebras for equational systems are also shown to be cocomplete and monadic over their base categories.

Chapter 5: Applications. This chapter illustrates the theory of equational systems with two sample modern applications: (i) π-calculus algebras of Stark [2005, 2008] and (ii) binding algebras with substitution structure of Fiore et al. [1999].

Part II. Term Equational Systems and Equational Reasoning.

Chapter 6: Term equational systems. We review the notion of action of a monoidal category (see *e.g.* [Janelidze and Kelly 2001]), which is used throughout Part II. Then we motivate and define the notion of term equational system and its model theory.

Chapter 7: Equational reasoning for term equational systems. A sound equational deduction system, called term equational logic, is proposed to reason about algebras for term equational systems. We also provide an internal completeness result and illustrate by an example that the internal completeness together with the inductive construction of free algebras developed in Chapter 3 may be used to synthesize a sound and complete rewriting-style equational logic for term equational systems.

Chapter 8: Applications. As substantial case studies, we derive complete deductive and rewriting-style equational logics for multi-sorted algebraic theories and nominal equational theories of Clouston and Pitts [2007] from our mathematical theory of term equational systems

The thesis concludes, in Chapter 9, with a brief summary and a discussion of related work and further research directions.

1.3.1 Published work

This thesis is largely based on the following articles written by the author with M. Fiore.

- **Equational systems and free constructions** [Fiore and Hur 2007].
- **Term equational systems and logics** [Fiore and Hur 2008].
- **On the construction of free algebras for equational systems** [Fiore and Hur 2009].

Part I is based on [Fiore and Hur 2009], which is a expanded version of [Fiore and Hur 2007]. Part II is based on [Fiore and Hur 2008], but has been significantly expanded.

Part I

Equational Systems and Free Constructions

Chapter 2

Equational systems

We introduce a general abstract notion of system of equations, called *equational system (ES)*, and its model theory. To motivate our definition of equational system, we start by reviewing the classical concept of algebraic theory (see *e.g.* [Wraith 1975, Crole 1994]) in Section 2.1. An algebraic theory is a system of terms and equations, given by a signature defining its operators and a set of equations describing the axioms that it should obey.

In Section 2.2, we define equational systems and their models. Generalizing the notion of signature for algebraic theories, we consider an endofunctor on a category as our abstract notion of signature, called *functorial signature*. The notion of term in context for a signature of an algebraic theory generalizes to that of *functorial term in functorial context* for a functorial signature. As usual, a pair of functorial terms in the same functorial context defines a *functorial equation*. An *equational system* is then given by a functorial signature conceptually defining its operators, and a functorial equation conceptually describing the axioms that it should satisfy. Models for equational systems naturally arise as algebras for their functorial signatures satisfying their functorial equations.

We introduce a notion of *monadic equational system*, which is a variant of equational system taking a monad as signature, and show that those systems can be turned into equational systems in such a way that models are preserved (see Section 2.3). In particular, *term equational systems* of Part II are represented as monadic equational systems. Also, the dual notion of equational system, called *equational cosystem*, is discussed (see Section 2.4). Finally, we present various examples, showing the expressiveness of equational systems (see Section 2.5).

2.1 Algebraic theories

We briefly review the classical concept of algebraic theory, and its model theory.

2.1.1 Signatures and equations

An algebraic theory consists of an algebraic signature, which specifies the operators allowed in the theory, and a set of equations, describing the axioms of the theory.

Definition 2.1.1. An *algebraic signature*, or just *signature*, $\Sigma = (\mathsf{O}, |-|)$ is given by a set of operators O together with a function $|-| : \mathsf{O} \to \mathbb{N}$ giving an arity to each operator.

Given a signature Σ specifying a set of operators, we can consider the notion of *term* on a set of variables V as follows: the set $T_\Sigma(V)$ of terms on V is built up from the variables and the operators of Σ by the following grammar

$$t \in T_\Sigma(V) \quad ::= \quad v \quad | \quad \mathsf{o}(t_1, \ldots, t_k)$$

where $v \in V$, o is an operator of arity k, and $t_i \in T_\Sigma(V)$ for $i = 1, \ldots, k$. An *equation* on a set V for a signature Σ, written $\Sigma \rhd V \vdash l \equiv r$, is simply defined as a pair of terms $l, r \in T_\Sigma(V)$.

Definition 2.1.2. An *algebraic theory* $\mathbb{T} = (\Sigma, E)$ is given by a signature Σ together with a set E of equations on finite sets.

We remark that the restriction that all equations of algebraic theories are on finite sets is without loss of generality. Indeed, every equation $\Sigma \rhd V \vdash l \equiv r$ can be turned into the equation $\Sigma \rhd \mathsf{Var}(l, r) \vdash l \equiv r$ on the finite set $\mathsf{Var}(l, r) \subseteq V$ consisting of the variables appearing in the terms l or r, in such a way that its model theoretic meaning is preserved. More precisely, using the notion of *satisfaction* to be introduced below, we have that a Σ-algebra satisfies $\Sigma \rhd V \vdash l \equiv r$ if and only if it satisfies $\Sigma \rhd \mathsf{Var}(l, r) \vdash l \equiv r$.

Example 2.1.3. As a running example, we consider the theory of groups $\mathbb{G} = (\Sigma_\mathbb{G}, E_\mathbb{G})$. The signature $\Sigma_\mathbb{G}$ consists of three operators: e of arity 0, i of arity 1, and m of arity 2, respectively corresponding to the three group operations: the identity, the inverse, and the multiplication. The set of equations $E_\mathbb{G}$ consists of the following equations representing the following group axioms:

$$\begin{array}{lcrcrcl}
\Sigma_\mathbb{G} & \rhd & \{\, x \,\} & \vdash & \mathsf{m}(x, \mathsf{e}) & = & x \\
\Sigma_\mathbb{G} & \rhd & \{\, x \,\} & \vdash & \mathsf{m}(x, \mathsf{i}(x)) & = & \mathsf{e} \\
\Sigma_\mathbb{G} & \rhd & \{\, x \,\} & \vdash & \mathsf{m}(\mathsf{i}(x), x) & = & \mathsf{e} \\
\Sigma_\mathbb{G} & \rhd & \{\, x, y, z \,\} & \vdash & \mathsf{m}(\mathsf{m}(x, y), z) & = & \mathsf{m}(x, \mathsf{m}(y, z)) \ .
\end{array}$$

2.1.2 Model theory

We now turn to the notion of model, called *algebra*, for algebraic theories.

Definition 2.1.4. An *algebra for a signature* Σ is a pair $(X, [\![-]\!])$ consisting of a carrier set X together with interpretation functions $[\![\mathsf{o}]\!] : X^{|\mathsf{o}|} \to X$ for each operator o in Σ.

A *homomorphism* of algebras for Σ from $(X, [\![-]\!])$ to $(Y, [\![-]\!]')$ is a function $h : X \to Y$ between their carrier sets that commutes with the interpretation of each operator; that is, such that $h([\![\mathsf{o}]\!](x_1, \ldots, x_k)) = [\![\mathsf{o}]\!]'(h(x_1), \ldots, h(x_k))$ for all $x_1, \ldots, x_k \in X$. Algebras and homomorphisms form the *category* Σ-**Alg** *of algebras for the signature* Σ.

By structural induction, such an algebra $(X, [\![-]\!])$ induces interpretations $[\![t]\!] : X^V \to X$ of terms $t \in T_\Sigma(V)$ as follows:

$$[\![t]\!] = \begin{cases} X^V \xrightarrow{\pi_v} X & \text{for } \ t = v \in V \\ X^V \xrightarrow{\langle [\![t_1]\!], \ldots, [\![t_k]\!] \rangle} X^k \xrightarrow{[\![\mathsf{o}]\!]} X & \text{for } \ t = \mathsf{o}(t_1, \ldots, t_k) \ . \end{cases} \tag{2.1}$$

A Σ-algebra $(X, [\![-]\!])$ is said to *satisfy* an equation $\Sigma \rhd V \vdash l \equiv r$ whenever the interpretations of the terms l and r coincide, *i.e.*, $[\![l]\!]\vec{x} = [\![r]\!]\vec{x}$ for all $\vec{x} \in X^V$.

Definition 2.1.5. An *algebra for a theory* $\mathbb{T} = (\Sigma, E)$ is an algebra for the signature Σ that satisfies every equation in E. The *category* $\mathbb{T}$-**Alg** *of algebras for the theory* $\mathbb{T}$ is the full subcategory of Σ-**Alg** consisting of the algebras for $\mathbb{T}$.

Example 2.1.6 (continued)**.** An algebra for the theory $\mathbb{G}$ is a set G equipped with operations $[\![\mathsf{e}]\!] : 1 \to G$, $[\![\mathsf{i}]\!] : G \to G$, $[\![\mathsf{m}]\!] : G^2 \to G$ satisfying the equations in $E_\mathbb{G}$:

$$\begin{array}{lrcl} \forall x \in G & [\![\mathsf{m}]\!](x, [\![\mathsf{e}]\!]()) & = & x \ , \\ \forall x \in G & [\![\mathsf{m}]\!](x, [\![\mathsf{i}]\!](x)) & = & [\![\mathsf{e}]\!]() \ , \\ \forall x \in G & [\![\mathsf{m}]\!]([\![\mathsf{i}]\!](x), x) & = & [\![\mathsf{e}]\!]() \ , \\ \forall x, y, z \in G & [\![\mathsf{m}]\!]([\![\mathsf{m}]\!](x, y), z) & = & [\![\mathsf{m}]\!](x, [\![\mathsf{m}]\!](y, z)) \ . \end{array}$$

This clearly defines a group in the usual sense. Also, homomorphisms between algebras for $\mathbb{G}$ coincide with group homomorphisms. Thus it follows that the category of algebras for $\mathbb{G}$ is (isomorphic to) the category of groups.

2.2 Equational systems

Generalizing the notions of signature, term and equation for algebraic theories, we develop abstract notions of signature, term and equation that lead to the concept of equational system.

2.2.1 Functorial signatures

We recall the well-known notion of algebra for an endofunctor and see how it generalizes that of algebra for an algebraic signature. This observation leads us to take endofunctors as our abstract notion of signature.

Definition 2.2.1 (Algebra for an endofunctor)**.** An *algebra for an endofunctor* Σ, or simply a Σ-*algebra*, on a category $\mathscr{C}$ is a pair (X, s) consisting of a *carrier object* X in $\mathscr{C}$ together with a *structure map* $s : \Sigma X \to X$. A *homomorphism* of Σ-algebras $(X, s) \to (Y, t)$ is a map $h : X \to Y$ in $\mathscr{C}$ such that $h \circ s = t \circ \Sigma h$. Σ-algebras and homomorphisms form the *category* Σ-**Alg**, and the *forgetful functor* $U_\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ maps a Σ-algebra (X, s) to its carrier object X.

Notation 2.2.2. For a Σ-algebra A, we use the notation $|A|$ for its carrier object and the notation $A^\diamond : \Sigma|A| \to |A|$ for its structure map; that is,

$$A = (|A|, A^\diamond)\,.$$

As it is well-known, every algebraic signature can be turned into an endofunctor on the category **Set** of sets and functions preserving its algebras. Indeed, for a signature Σ, one defines the corresponding endofunctor F_Σ by

$$F_\Sigma(X) = \coprod_{\mathsf{o} \in \Sigma} X^{|\mathsf{o}|}\,,$$

so that Σ-**Alg** and F_Σ-**Alg** are isomorphic. In this view, we will henceforth take endofunctors as a general abstract notion of signature.

Definition 2.2.3 (Functorial signature)**.** A *functorial signature* on a category is an endofunctor on it.

Example 2.2.4 (continued)**.** For the theory $\mathbb{G}$ of groups, the functorial signature $F_{\Sigma_\mathbb{G}}$ on **Set** is defined by

$$F_{\Sigma_\mathbb{G}}(X) = 1 + X + X^2\,.$$

As an $F_{\Sigma_\mathbb{G}}$-algebra is given by $(X, [[\![\mathsf{e}]\!], [\![\mathsf{i}]\!], [\![\mathsf{m}]\!]] : 1 + X + X^2 \to X)$, the notion of $F_{\Sigma_\mathbb{G}}$-algebra is equivalent to that of algebra for the algebraic signature $\Sigma_\mathbb{G}$.

2.2.2 Functorial terms and equations

We motivate and present abstract notions of term and equation for functorial signatures.

Let $t \in T_\Sigma(V)$ be a term on a set of variables V for a signature Σ. Recall from the previous section that for every Σ-algebra $(X, [\![-]\!])$, the term t gives an interpretation function $[\![t]\!] : X^V \to X$. Thus, writing Γ_V for the endofunctor $(-)^V$ on **Set**, the term t determines a function $\bar{t}$ assigning to a Σ-algebra $(X, [\![-]\!])$ the Γ_V-algebra $(X, [\![t]\!])$. Note that the function $\bar{t}$ does not only preserve carrier objects but, furthermore, by the uniformity of the interpretation of terms, satisfies that a Σ-homomorphism $(X, [\![-]\!]) \to (Y, [\![-]\!]')$ is also a Γ_V-homomorphism $(X, [\![t]\!]) \to (Y, [\![t]\!]')$. In other words, the function $\bar{t}$ extends to a functor $\Sigma\text{-}\mathbf{Alg} \to \Gamma_V\text{-}\mathbf{Alg}$ over **Set**, *i.e.*, a functor preserving carrier objects and homomorphisms. These considerations lead us to define an abstract notion of term in context as follows.

Definition 2.2.5 (Functorial term and equation)**.** A *functorial term* T in a *functorial context* Γ for a functorial signature Σ on a category $\mathscr{C}$, denoted $\mathscr{C} : \Sigma \rhd \Gamma \vdash T$, is given by an endofunctor Γ on $\mathscr{C}$ and a functor $T : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ over $\mathscr{C}$; that is, a functor such that $U_\Gamma \circ T = U_\Sigma$. A *functorial equation* $\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R$ is a pair of functorial terms L and R in the same context Γ.

Example 2.2.6 (continued)**.** The equations of the theory $\mathbb{G}$ of groups induce the following functorial equations for the functorial signature $F_{\Sigma_\mathbb{G}}$:

$$
\begin{array}{llrlrcl}
\mathbf{Set} : F_{\Sigma_\mathbb{G}} & \rhd & [\![\{x\}]\!] & \vdash & [\![\mathsf{m}(x,\mathsf{e})]\!] & \equiv & [\![x]\!] \\
\mathbf{Set} : F_{\Sigma_\mathbb{G}} & \rhd & [\![\{x\}]\!] & \vdash & [\![\mathsf{m}(x,\mathsf{i}(x))]\!] & \equiv & [\![\mathsf{e}]\!] \\
\mathbf{Set} : F_{\Sigma_\mathbb{G}} & \rhd & [\![\{x\}]\!] & \vdash & [\![\mathsf{m}(\mathsf{i}(x),x)]\!] & \equiv & [\![\mathsf{e}]\!] \\
\mathbf{Set} : F_{\Sigma_\mathbb{G}} & \rhd & [\![\{x,y,z\}]\!] & \vdash & [\![\mathsf{m}(\mathsf{m}(x,y),z)]\!] & \equiv & [\![\mathsf{m}(x,\mathsf{m}(y,z))]\!]
\end{array}
$$

where the functorial contexts are defined by setting, for every $X \in \mathbf{Set}$,

$$
[\![\{x\}]\!]\,(X) \;=\; X\,, \qquad\qquad [\![\{x,y,z\}]\!]\,(X) \;=\; X^3
$$

and where the functorial terms are defined by setting,

for every $F_{\Sigma_\mathbb{G}}$-algebra $(X, [[\![\mathsf{e}]\!], [\![\mathsf{i}]\!], [\![\mathsf{m}]\!]] : 1 + X + X^2 \to X)$,

$$
\begin{array}{rcl}
[\![\mathsf{m}(x,\mathsf{e})]\!]\,(X,[[\![\mathsf{e}]\!],[\![\mathsf{i}]\!],[\![\mathsf{m}]\!]]) & = & (X,\ X \xrightarrow{\langle \mathrm{id}_X\,,\,[\![\mathsf{e}]\!]\circ !_X\rangle} X^2 \xrightarrow{[\![\mathsf{m}]\!]} X) \\
{[\![x]\!]}\,(X,[[\![\mathsf{e}]\!],[\![\mathsf{i}]\!],[\![\mathsf{m}]\!]]) & = & (X,\ X \xrightarrow{\mathrm{id}_X} X) \\
{[\![\mathsf{m}(x,\mathsf{i}(x))]\!]}\,(X,[[\![\mathsf{e}]\!],[\![\mathsf{i}]\!],[\![\mathsf{m}]\!]]) & = & (X,\ X \xrightarrow{\langle \mathrm{id}_X\,,\,[\![\mathsf{i}]\!]\rangle} X^2 \xrightarrow{[\![\mathsf{m}]\!]} X) \\
{[\![\mathsf{m}(\mathsf{i}(x),x)]\!]}\,(X,[[\![\mathsf{e}]\!],[\![\mathsf{i}]\!],[\![\mathsf{m}]\!]]) & = & (X,\ X \xrightarrow{\langle [\![\mathsf{i}]\!]\,,\,\mathrm{id}_X\rangle} X^2 \xrightarrow{[\![\mathsf{m}]\!]} X) \\
{[\![\mathsf{e}]\!]}\,(X,[[\![\mathsf{e}]\!],[\![\mathsf{i}]\!],[\![\mathsf{m}]\!]]) & = & (X,\ X \xrightarrow{[\![\mathsf{e}]\!]\circ !_X} X) \\
{[\![\mathsf{m}(\mathsf{m}(x,y),z)]\!]}\,(X,[[\![\mathsf{e}]\!],[\![\mathsf{i}]\!],[\![\mathsf{m}]\!]]) & = & (X,\ X^3 \xrightarrow{\langle [\![\mathsf{m}]\!]\circ\langle\pi_1,\pi_2\rangle\,,\,\pi_3\rangle} X^2 \xrightarrow{[\![\mathsf{m}]\!]} X) \\
{[\![\mathsf{m}(x,\mathsf{m}(y,z))]\!]}\,(X,[[\![\mathsf{e}]\!],[\![\mathsf{i}]\!],[\![\mathsf{m}]\!]]) & = & (X,\ X^3 \xrightarrow{\langle \pi_1\,,\,[\![\mathsf{m}]\!]\circ\langle\pi_2,\pi_3\rangle\rangle} X^2 \xrightarrow{[\![\mathsf{m}]\!]} X)
\end{array}
$$

where the map $!_X : X \to 1$ is the unique map to the terminal object 1.

As we have seen from the example of the theory of groups, the intuitions behind functorial signatures, contexts and terms are summarised as follows.

- A functorial signature Σ represents a set of operators, and a Σ-algebra $(X, s : \Sigma X \to X)$ gives interpretations to the operators.
- A functorial context Γ represents a set of variables and the object ΓX consists of all valuations of the variables in X.

- A functorial term $\mathscr{C} : \Sigma \rhd \Gamma \vdash T$ represents a term built up from the operators of Σ and the variables of Γ, and the functor $T : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ amounts to the process of evaluating the term to a value, parametrically on interpretations of the operators and valuations of the variables.

2.2.3 Equational systems

We define equational systems, our abstract notion of system of equations.

Definition 2.2.7 (Equational system)**.** An *equational system*

$$\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$$

is given by a functorial equation $\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R$ for a functorial signature Σ on a category $\mathscr{C}$.

We have restricted attention to equational systems subject to a single equation. The consideration of multi-equational systems ($\mathscr{C} : \Sigma \rhd \{\Gamma_i \vdash L_i \equiv R_i\}_{i\in I}$) subject to a set of equations in what follows is left to the interested reader. We remark however that our development is typically without loss of generality; as, whenever $\mathscr{C}$ has I-indexed coproducts, a multi-equational system as above can be expressed as the equational system $(\mathscr{C} : \Sigma \rhd \coprod_{i\in I} \Gamma_i \vdash [L_i]_{i\in I} \equiv [R_i]_{i\in I})$ with a single equation, where the functorial context and terms are defined by

$$\begin{aligned}
(\textstyle\coprod_{i\in I} \Gamma_i)(X) &\triangleq \textstyle\coprod_{i\in I} \Gamma_i(X) \ , \\
[L_i]_{i\in I}(X, s) &\triangleq (X, [L_i(X, s)^\diamond]_{i\in I} : \textstyle\coprod_{i\in I} \Gamma_i(X) \longrightarrow X) \ , \\
[R_i]_{i\in I}(X, s) &\triangleq (X, [R_i(X, s)^\diamond]_{i\in I} : \textstyle\coprod_{i\in I} \Gamma_i(X) \longrightarrow X) \ .
\end{aligned}$$

We now consider algebras for equational systems. To this end, recall that an algebra for an algebraic signature Σ is said to satisfy an equation $\Sigma \rhd V \vdash l \equiv r$ when the interpretation functions associated to the terms l and r coincide. Hence, it is natural to say that a Σ-algebra (X, s) satisfies a functorial equation $\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R$ whenever $L(X, s)^\diamond = R(X, s)^\diamond : \Gamma X \to X$. This consideration induces the following definition of algebras for equational systems.

Definition 2.2.8 (Algebra for an equational system)**.** An *algebra for an equational system* $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$, or simply an $\mathbb{S}$-*algebra*, is a Σ-algebra (X, s) *satisfying* the functorial equation $\Gamma \vdash L \equiv R$; that is, such that $L(X, s)^\diamond = R(X, s)^\diamond : \Gamma X \to X$. The category $\mathbb{S}$-$\mathbf{Alg}$ is the full subcategory of Σ-$\mathbf{Alg}$ consisting of $\mathbb{S}$-algebras.

Note that the category $\mathbb{S}$-$\mathbf{Alg}$ of $\mathbb{S}$-algebras is an equalizer of $L, R : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ in the large category $\mathbf{CAT}/\mathscr{C}$ of locally small categories over $\mathscr{C}$, *i.e.*, the large category with objects given by pairs $(\mathscr{E}, U : \mathscr{E} \to \mathscr{C})$ consisting of a locally small category $\mathscr{E}$ and a functor $U : \mathscr{E} \to \mathscr{C}$; and with morphisms $(\mathscr{E}, U) \to (\mathscr{E}', U')$ given by functors $F : \mathscr{E} \to \mathscr{E}'$ such that $U' F = U$.

We now organize equational systems into a category.

Definition 2.2.9 (Category of equational systems)**.** The *category* $\mathbf{ES}(\mathscr{C})$ *of equational systems on a category* $\mathscr{C}$ has objects given by equational systems on $\mathscr{C}$ and morphisms $\mathbb{S} \to \mathbb{S}'$ given by functors $\mathbb{S}'\text{-}\mathbf{Alg} \to \mathbb{S}\text{-}\mathbf{Alg}$ preserving carrier objects and homomorphisms.

Note that the category of equational systems on a category $\mathscr{C}$ forms a full subcategory of $(\mathbf{CAT}/\mathscr{C})^{\mathrm{op}}$ through the embedding $\mathbf{ES}(\mathscr{C}) \hookrightarrow (\mathbf{CAT}/\mathscr{C})^{\mathrm{op}}$ sending an equational system $\mathbb{S}$ to the pair $(\mathbb{S}\text{-}\mathbf{Alg}, U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C})$. The definition of the category of equational systems is consistent with that for algebraic theories. Indeed, the category of algebraic theories (see *e.g.* [Wraith 1975, Section 5], [Crole 1994, Discussion 3.9.5]) appears as a full subcategory of the category $\mathbf{ES}(\mathbf{Set})$ of equational systems on $\mathbf{Set}$ through the encoding of algebraic theories into equational systems, to be presented in item 1 of Section 2.5. Also, the category $\mathbf{Mnd}(\mathscr{C})$ of monads on a category $\mathscr{C}$ with binary coproducts appears as a full subcategory of $\mathbf{ES}(\mathscr{C})$ through the encoding of monads into equational systems, to be given in item 3 of Section 2.5.

Example 2.2.10 (continued)**.** The equational system $\mathbb{S}_{\mathbb{G}}$ of groups is defined by

$$
\begin{array}{rcllllllll}
\mathbb{S}_{\mathbb{G}} & = & (& \mathbf{Set} : F_{\Sigma_{\mathbb{G}}} \\
& & \rhd & [\![\{x\}]\!] & + & [\![\{x\}]\!] & + & [\![\{x\}]\!] & + & [\![\{x,y,z\}]\!] \\
& & \vdash & [\ [\![\mathsf{m}(x,\mathsf{e})]\!] & , & [\![\mathsf{m}(x,\mathsf{i}(x))]\!] & , & [\![\mathsf{m}(\mathsf{i}(x),x)]\!] & , & [\![\mathsf{m}(\mathsf{m}(x,y),z)]\!]\ ] \\
& & \equiv & [\ [\![x]\!] & , & [\![\mathsf{e}]\!] & , & [\![\mathsf{e}]\!] & , & [\![\mathsf{m}(x,\mathsf{m}(y,z))]\!]\ ]\ ) .
\end{array}
$$

It follows that $\mathbb{S}_{\mathbb{G}}\text{-}\mathbf{Alg}$ is isomorphic to the category of algebras for the theory $\mathbb{G}$; that is, the category of groups.

2.3 Monadic equational systems

We introduce a notion of monadic equational system and provide an encoding of these systems into equational systems in such a way that models are preserved. Note that *term equational systems* of Part II will be main examples of monadic equational systems.

Definition 2.3.1. A *monadic equational system* $\mathbb{S} = (\mathscr{C} : \mathbf{T} \rhd \Gamma \vdash L \equiv R)$ is given by a category $\mathscr{C}$, a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathscr{C}$, an endofunctor Γ on $\mathscr{C}$, and a pair of functors $L, R : \mathscr{C}^{\mathbf{T}} \to \Gamma\text{-}\mathbf{Alg}$ preserving carrier objects and homomorphisms, for $\mathscr{C}^{\mathbf{T}}$ the category of Eilenberg-Moore algebras of the monad $\mathbf{T}$. An $\mathbb{S}$-algebra is an Eilenberg-Moore algebra (X, s) for $\mathbf{T}$ satisfying $L(X,s)^{\diamond} = R(X,s)^{\diamond} : \Gamma X \to X$. The category $\mathbb{S}\text{-}\mathbf{Alg}$ is the full subcategory of $\mathscr{C}^{\mathbf{T}}$ consisting of $\mathbb{S}$-*algebras.*

When $\mathscr{C}$ has binary coproducts, the monadic system $\mathbb{S}$ can be encoded as the equational system

$$\overline{\mathbb{S}} = (\mathscr{C} : T \rhd (\Gamma_{\mathbf{T}} + \Gamma) \vdash [L_{\mathbf{T}}, \overline{L}] \equiv [R_{\mathbf{T}}, \overline{R}])$$

where $\Gamma_{\mathbf{T}}(X) = X + TTX$ and, for all T-algebras (X, s),

$$\begin{aligned}
L_{\mathbf{T}}(X,s) &= (\, X,\ [\ \ s \circ \eta_X \ \ ,\ \ s \circ \mu_X \ \ ]\,)\,,\\
R_{\mathbf{T}}(X,s) &= (\, X,\ [\ \ \mathrm{id}_X \ \ ,\ \ s \circ Ts \ \ ]\,)\,,\\
\overline{L}(X,s) &= (\, X,\ \Gamma X \xrightarrow{\Gamma\eta_X} \Gamma TX \xrightarrow{L(TX,\mu_X)^\diamond} TX \xrightarrow{s} X\,)\,,\\
\overline{R}(X,s) &= (\, X,\ \Gamma X \xrightarrow{\Gamma\eta_X} \Gamma TX \xrightarrow{R(TX,\mu_X)^\diamond} TX \xrightarrow{s} X\,)\,.
\end{aligned}$$

One can easily see that the categories $\mathbb{S}$-**Alg** and $\overline{\mathbb{S}}$-**Alg** coincide from the following observations. An $\overline{\mathbb{S}}$-algebra is a T-algebra (X, s) satisfying (i) $L_{\mathbf{T}}(X,s)^\diamond = R_{\mathbf{T}}(X,s)^\diamond$ and (ii) $\overline{L}(X,s)^\diamond = \overline{R}(X,s)^\diamond$. The condition (i) states that (X, s) is an Eilenberg-Moore algebra for the monad $\mathbf{T}$. For an Eilenberg-Moore algebra $(X,s) \in \mathscr{C}^{\mathbf{T}}$, the condition (ii) is equivalent to the condition $L(X,s)^\diamond = R(X,s)^\diamond$ because $\overline{L}(X,s)^\diamond = L(X,s)^\diamond$ and $\overline{R}(X,s)^\diamond = R(X,s)^\diamond$, as indicated by the following commutative diagrams

$$\begin{tikzcd}
\Gamma X \arrow[r, "\Gamma\eta_X"] \arrow[rr, bend left, "\mathrm{id}"] & \Gamma TX \arrow[r, "\Gamma s"] \arrow[d, "L(TX,\mu_X)^\diamond"'] \arrow[dr, phantom, "\mathsf{(A)}"] & \Gamma X \arrow[d, "L(X,s)^\diamond"] \\
& TX \arrow[r, "s"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma X \arrow[r, "\Gamma\eta_X"] \arrow[rr, bend left, "\mathrm{id}"] & \Gamma TX \arrow[r, "\Gamma s"] \arrow[d, "R(TX,\mu_X)^\diamond"'] \arrow[dr, phantom, "\mathsf{(B)}"] & \Gamma X \arrow[d, "R(X,s)^\diamond"] \\
& TX \arrow[r, "s"] & X
\end{tikzcd}$$

where the diagrams (A) and (B) commute because $s : TX \to X$ is a homomorphism from (TX, μ_X) to (X, s) in the category $\mathscr{C}^{\mathbf{T}}$.

The situation is summarised as follows:

$$\begin{tikzcd}
& & \Gamma\text{-}\mathbf{Alg} & \Gamma_{\mathbf{T}}\text{-}\mathbf{Alg} \\
\mathbb{S}\text{-}\mathbf{Alg} = \overline{\mathbb{S}}\text{-}\mathbf{Alg} \arrow[r, hook, "J"'] & \mathscr{C}^{\mathbf{T}} \arrow[r, hook, "J_{\mathbf{T}}"'] \arrow[ur, shift left, "L"] \arrow[ur, shift right, "R"'] & T\text{-}\mathbf{Alg} \arrow[u, shift left, "\overline{L}"] \arrow[u, shift right, "\overline{R}"'] \arrow[ur, shift left, "L_{\mathbf{T}}"] \arrow[ur, shift right, "R_{\mathbf{T}}"'] &
\end{tikzcd}$$

where $J_{\mathbf{T}}$ and J are the canonical embeddings, and we have that (i) $J_{\mathbf{T}}$ is an equalizer of $L_{\mathbf{T}}, R_{\mathbf{T}}$; that (ii) $L = \overline{L}\,J_{\mathbf{T}}$ and $R = \overline{R}\,J_{\mathbf{T}}$; and that (iii) J is an equalizer of L, R.

2.4 Equational cosystems

The usual notion of coalgebra for an endofunctor Σ on a category $\mathscr{C}$ arises as that of algebra for the endofunctor Σ^{op} on the opposite category $\mathscr{C}^{\mathrm{op}}$. Similarly, the concept of equational system is dualized by considering an *equational cosystem* on $\mathscr{C}$ to be an equational system on $\mathscr{C}^{\mathrm{op}}$. Consequently all the results we obtain for equational systems in the subsequent chapters apply to equational cosystems in their dual versions.

The above consideration dualizes all notions for equational systems as follows.

Definition 2.4.1. A *coalgebra for an endofunctor* Σ on a category $\mathscr{C}$ is a pair (X, s) consisting of a *carrier object* X in $\mathscr{C}$ together with a *structure map* $s : X \to \Sigma X$. A *homomorphism* of Σ-coalgebras $(X, s) \to (Y, t)$ is a map $h : X \to Y$ in $\mathscr{C}$ such that $\Sigma h \circ s = t \circ h$. Σ-coalgebras and homomorphisms form the *category* Σ-**CoAlg**, and the *forgetful functor* $U_\Sigma : \Sigma\text{-}\mathbf{CoAlg} \to \mathscr{C}$ maps a Σ-coalgebra (X, s) to its carrier object X.

Notation 2.4.2. For a Σ-coalgebra A, we also denote its carrier object and structure map by $|A|$ and $A^\diamond$ respectively.

Definition 2.4.3. A *functorial cosignature* Σ on a category $\mathscr{C}$ is an endofunctor on it. A *functorial coterm* T in a *functorial cocontext* Γ for the functorial cosignature Σ on $\mathscr{C}$, denoted $\mathscr{C} : \Sigma \rhd \Gamma \vdash T$, is given by an endofunctor Γ on $\mathscr{C}$ and a functor $T : \Sigma\text{-}\mathbf{CoAlg} \to \Gamma\text{-}\mathbf{CoAlg}$ such that $U_\Gamma \circ T = U_\Sigma$.

Definition 2.4.4. An *equational cosystem* $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ consists of a category $\mathscr{C}$ together with a pair of functorial coterms $\mathscr{C} : \Sigma \rhd \Gamma \vdash L$ and $\mathscr{C} : \Sigma \rhd \Gamma \vdash R$, referred to as a *functorial coequation.* An $\mathbb{S}$-*coalgebra* (X, s) is a Σ-coalgebra satisfying the equation; that is, such that $L(X, s)^\diamond = R(X, s)^\diamond : X \to \Gamma X$. The category $\mathbb{S}$-**CoAlg** is the full subcategory of Σ-**CoAlg** consisting of $\mathbb{S}$-coalgebras, and the *forgetful functor* $U_\mathbb{S} : \mathbb{S}\text{-}\mathbf{CoAlg} \to \mathscr{C}$ maps $\mathbb{S}$-coalgebras to their carrier objects.

Definition 2.4.5. The *category* $\mathbf{ECoS}(\mathscr{C})$ *of equational cosystems on a category* $\mathscr{C}$ has objects given by equational cosystems on $\mathscr{C}$ and morphisms $\mathbb{S} \to \mathbb{S}'$ given by functors $\mathbb{S}\text{-}\mathbf{CoAlg} \to \mathbb{S}'\text{-}\mathbf{CoAlg}$ preserving carrier objects and homomorphisms.

The relation between equational cosystems and equational systems is formalized as follows. Every endofunctor Σ on a category $\mathscr{C}$ bijectively maps to the opposite endofunctor Σ^{op} on the opposite category $\mathscr{C}^{\mathrm{op}}$; and it follows that

$$\begin{array}{rcll} (\Sigma\text{-}\mathbf{CoAlg})^{\mathrm{op}} & = & \Sigma^{\mathrm{op}}\text{-}\mathbf{Alg} & \\ (U_\Sigma)^{\mathrm{op}} & = & U_{\Sigma^{\mathrm{op}}} & : \ (\Sigma\text{-}\mathbf{CoAlg})^{\mathrm{op}} = \Sigma^{\mathrm{op}}\text{-}\mathbf{Alg} \longrightarrow \mathscr{C}^{\mathrm{op}} . \end{array}$$

Similarly, every equational cosystem $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ on a category $\mathscr{C}$ bijectively maps to the opposite equational system $\mathbb{S}^{\mathrm{op}}$ on the opposite category $\mathscr{C}^{\mathrm{op}}$ given by

$$\mathbb{S}^{\mathrm{op}} \; = \; (\mathscr{C}^{\mathrm{op}} : \Sigma^{\mathrm{op}} \rhd \Gamma^{\mathrm{op}} \vdash L^{\mathrm{op}} \equiv R^{\mathrm{op}}) \, ;$$

and it follows that

$$\begin{array}{rcll} (\mathbb{S}\text{-}\mathbf{CoAlg})^{\mathrm{op}} & = & \mathbb{S}^{\mathrm{op}}\text{-}\mathbf{Alg} & \\ (U_\mathbb{S})^{\mathrm{op}} & = & U_{\mathbb{S}^{\mathrm{op}}} & : \ (\mathbb{S}\text{-}\mathbf{CoAlg})^{\mathrm{op}} = \mathbb{S}^{\mathrm{op}}\text{-}\mathbf{Alg} \longrightarrow \mathscr{C}^{\mathrm{op}} . \end{array}$$

Finally we have the following isomorphism through the above bijection between equational cosystems on $\mathscr{C}$ and equational systems on $\mathscr{C}^{\mathrm{op}}$:

$$\big(\mathbf{ECoS}(\mathscr{C})\big)^{\mathrm{op}} \; \cong \; \mathbf{ES}(\mathscr{C}^{\mathrm{op}}) \, .$$

For an example of equational cosystem, see item 5 of Section 2.5.

2.5 Examples

We show the expressiveness of the notion of equational system by encoding various systems of equations as equivalent equational systems in the sense that their categories of algebras coincide.

1. The equational system $\mathbb{S}_\mathbb{T}$ associated to an algebraic theory $\mathbb{T} = (\Sigma, E)$ is given by $(\mathbf{Set} : \Sigma_\mathbb{T} \rhd \Gamma_\mathbb{T} \vdash L_\mathbb{T} \equiv R_\mathbb{T})$ with

$$\begin{aligned} \Sigma_\mathbb{T} X &= \textstyle\coprod_{\mathsf{o}\in\Sigma} X^{|\mathsf{o}|} \ , \\ \Gamma_\mathbb{T} X &= \textstyle\coprod_{(V\vdash l\equiv r)\in E} X^{V} \ , \\ L_\mathbb{T}(X, [\,[\![\mathsf{o}]\!]\,]_{\mathsf{o}\in\Sigma}) &= \big(X, [\,[\![l]\!]\,]_{(V\vdash l\equiv r)\in E}\big) \ , \\ R_\mathbb{T}(X, [\,[\![\mathsf{o}]\!]\,]_{\mathsf{o}\in\Sigma}) &= \big(X, [\,[\![r]\!]\,]_{(V\vdash l\equiv r)\in E}\big) \ . \end{aligned}$$

It follows that $\mathbb{S}_\mathbb{T}$-$\mathbf{Alg}$ is (isomorphic to) the category $\mathbb{T}$-$\mathbf{Alg}$ of algebras for $\mathbb{T}$.

2. Recall the notion of *enriched algebraic theory* from Introduction of the thesis (see Section 1.1.2). Though the notion of base category for enriched algebraic theories was considered in restricted form in Introduction, we consider it here in general form (see [Kelly and Power 1993, Robinson 2002]).

Consider an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, \Sigma, E)$ given by a base category $\mathscr{C}$ (which is a locally finitely presentable category enriched over a symmetric monoidal closed category $\mathscr{V}$ that is locally finitely presentable as a closed category), a signature Σ (which is a set of operators with arities and coarities) and a set E of equations. The equational system $\mathbb{S}_\mathbb{T}$ associated to the enriched algebraic theory $\mathbb{T}$ is given by $(\mathscr{C}_0 : \Sigma_\mathbb{T} \rhd \Gamma_\mathbb{T} \vdash L_\mathbb{T} \equiv R_\mathbb{T})$ with

$$\begin{aligned} \Sigma_\mathbb{T} X &= \textstyle\coprod_{\mathsf{o}\in\Sigma \text{ with arity } A, \text{ coarity } C} \mathscr{C}(A,X)\otimes C \ , \\ \Gamma_\mathbb{T} X &= \textstyle\coprod_{(l\equiv r)\in E \text{ with arity } A, \text{ coarity } C} \mathscr{C}(A,X)\otimes C \ , \\ L_\mathbb{T}(X, [\,[\![\mathsf{o}]\!]\,]_{\mathsf{o}\in\Sigma}) &= \big(X, [\,[\![l]\!]\,]_{(l\equiv r)\in E}\big) \ , \\ R_\mathbb{T}(X, [\,[\![\mathsf{o}]\!]\,]_{\mathsf{o}\in\Sigma}) &= \big(X, [\,[\![r]\!]\,]_{(l\equiv r)\in E}\big) \end{aligned}$$

where $\mathscr{C}_0$ denotes the underlying ordinary category of the enriched category $\mathscr{C}$, $\mathscr{C}(-,=) : \mathscr{C}_0 \times \mathscr{C}_0 \to \mathscr{V}$ is the hom-functor of the enriched category $\mathscr{C}$ and $(-)\otimes(=) : \mathscr{V}\times\mathscr{C}_0 \to \mathscr{C}_0$ is the tensor of the enrichment; and where the interpretation map $[\![t]\!]$ of a term $t : C \to \mathbf{T}_\Sigma A$ for a $\Sigma_\mathbb{T}$-algebra $(X, [\,[\![\mathsf{o}]\!]\,]_{\mathsf{o}\in\Sigma})$ is given by the composite

$$\mathscr{C}(A,X)\otimes C \xrightarrow{\mathscr{C}(A,X)\otimes t} \mathscr{C}(A,X)\otimes \mathbf{T}_\Sigma A \xrightarrow{\mathsf{st}_{\mathscr{C}(A,X),A}} \mathbf{T}_\Sigma(\mathscr{C}(A,X)\otimes A) \xrightarrow{\mathbf{T}_\Sigma(\epsilon)} \mathbf{T}_\Sigma X \xrightarrow{\overline{[\![\cdot]\!]}} X$$

for $(\mathbf{T}_\Sigma, \mathsf{st})$ the strong monad induced from the signature Σ and $(X, \overline{[\![\cdot]\!]})$ the Eilenberg-Moore algebra for $\mathbf{T}_\Sigma$ corresponding to the $\Sigma_\mathbb{T}$-algebra $(X, [\,[\![\mathsf{o}]\!]\,]_{\mathsf{o}\in\Sigma})$. It follows that $\mathbb{S}_\mathbb{T}$-$\mathbf{Alg}$ is (isomorphic to) the ordinary category $(\mathbb{T}\text{-}\mathbf{Alg})_0$ of algebras for $\mathbb{T}$.

3. Eilenberg-Moore algebras for a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ with binary coproducts can be easily turned into algebras for an equational system, as they are algebras for the monadic equational system on $\mathscr{C}$ with the monad $\mathbf{T}$ as signature and with no equation (technically, with a tautological equation). Thus, the equational system $\mathbb{S}_{\mathbf{T}}$ representing the monad $\mathbf{T}$ is given by $(\mathscr{C} : T \rhd \Gamma_{\mathbf{T}} \vdash L_{\mathbf{T}} \equiv R_{\mathbf{T}})$ with

$$\begin{aligned} \Gamma_{\mathbf{T}}(X) &= X + TTX \;, \\ L_{\mathbf{T}}(X, s) &= (\,X, [\; s \circ \eta_X \;,\; s \circ \mu_X \;]\,) \;, \\ R_{\mathbf{T}}(X, s) &= (\,X, [\; \mathrm{id}_X \;,\; s \circ Ts \;]\,) \;. \end{aligned}$$

It follows that $\mathbb{S}_{\mathbf{T}}$-**Alg** is (isomorphic to) the category $\mathscr{C}^{\mathbf{T}}$ of Eilenberg-Moore algebras for the monad $\mathbf{T}$.

4. The definition of monoid in a monoidal category $(\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ with binary coproducts yields the equational system $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ with

$$\begin{aligned} \Sigma(X) &= I + (X \otimes X) \;, \\ \Gamma(X) &= (I \otimes X) + (X \otimes I) + \big((X \otimes X) \otimes X\big) \;, \\ L(X, [e, m]) &= (\,X, [\; \lambda_X \;,\; \rho_X \;,\; m \circ (m \otimes \mathrm{id}_X) \;]\,) \,, \\ R(X, [e, m]) &= (\,X, [\; m \circ (e \otimes \mathrm{id}_X) \;,\; m \circ (\mathrm{id}_X \otimes e) \;,\; m \circ (\mathrm{id}_X \otimes m) \circ \alpha_{X,X,X} \;]\,) \,. \end{aligned}$$

It follows that $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$-**Alg** is (isomorphic to) the category of monoids and monoid homomorphisms in $\mathscr{C}$.

5. The definition of comonoid in a monoidal category $(\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ with binary products yields the equational cosystem $\mathbb{S}_{\mathrm{CoMon}(\mathscr{C})} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ with

$$\begin{aligned} \Sigma(X) &= I \times (X \otimes X) \\ \Gamma(X) &= (I \otimes X) \times (X \otimes I) \times \big((X \otimes X) \otimes X\big) \\ L(X, \langle e, m \rangle) &= (\,X, \big\langle\; \lambda_X^{-1} \;,\; \rho_X^{-1} \;,\; (m \otimes \mathrm{id}_X) \circ m \;\big\rangle\,) \,, \\ R(X, \langle e, m \rangle) &= (\,X, \big\langle\; (e \otimes \mathrm{id}_X) \circ m \;,\; (\mathrm{id}_X \otimes e) \circ m \;,\; \alpha_{X,X,X}^{-1} \circ (\mathrm{id}_X \otimes m) \circ m \;\big\rangle\,) \,. \end{aligned}$$

It follows that $\mathbb{S}_{\mathrm{CoMon}(\mathscr{C})}$-**CoAlg** is (isomorphic to) the category of comonoids and comonoid homomorphisms in $\mathscr{C}$.

Chapter 3

Theory of inductive equational systems

In this and the next chapter, we study properties of equational systems. To motivate the properties that we are interested in, we briefly review some well-known properties for algebraic theories: namely, the existence of free algebras, and the monadicity and cocompleteness of categories of algebras (see Section 3.1). Unlike for algebraic theories, however, these properties do not hold for all equational systems and thus we seek sufficient conditions for these properties to hold. Throughout this chapter, we concentrate on a simple and practical condition, called *inductiveness*. More general conditions are studied in the next chapter.

Definition 3.0.1. A functor is called *epicontinuous* if it preserves epimorphisms, and called *κ-cocontinuous* for an ordinal κ if it preserves colimits of κ-chains.

Definition 3.0.2. An equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ is said to be *inductive* if the category $\mathscr{C}$ is cocomplete and the endofunctors Σ and Γ on $\mathscr{C}$ are epicontinuous and ω-cocontinuous.

In Section 3.2, we present an explicit categorical construction of free algebras for inductive equational systems which directly generalizes that for algebraic theories. More precisely, the construction of a free $\mathbb{S}$-algebra on an object V in $\mathscr{C}$, for an inductive equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$, captures the following intuition.

1. An object $T_\Sigma V$, intuitively consisting of terms built up from operators in Σ and variables in V, is inductively constructed.

2. The object $T_\Sigma V$ is first quotiented by the equation $\Gamma \vdash L \equiv R$ and then iteratively quotiented by congruence rules for operators in Σ, to obtain a free $\mathbb{S}$-algebra on V.

In Section 3.3, we show other properties of (inductive) equational systems. For this, we define the notion of representing monad for equational systems.

Definition 3.0.3. For an equational system $\mathbb{S}$ on a category $\mathscr{C}$ for which the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, the associated monad is called the *representing monad* of the system $\mathbb{S}$.

This terminology is justified by the fact that the adjunction is always monadic (see Proposition 3.3.4). Then we show that, for every inductive equational system $\mathbb{S}$,

- the category $\mathbb{S}$-**Alg** of $\mathbb{S}$-algebras is cocomplete and monadic over the base category; and
- the representing monad of $\mathbb{S}$ is epicontinuous and ω-cocontinuous.

We also show that

- the category **IndES**$(\mathscr{C})$ of inductive equational systems on a cocomplete category $\mathscr{C}$ is cocomplete.

In Section 3.4, we conclude the chapter discussing the properties of the example equational systems given in Section 2.5 in the light of the above results.

3.1 Free constructions and properties for algebraic theories

We quickly review the well-known construction of free algebras for algebraic theories, and some known properties of algebraic theories (see *e.g.* [Wraith 1975, Crole 1994, Pedicchio and Tholen 2004]).

For an algebraic theory $\mathbb{T} = (\Sigma, E)$, the free algebra over a set of variables V has as carrier the quotient set $T_\Sigma(V)/_{\approx_E}$ consisting of equivalence classes of terms on V under the relation $\approx_E$ generated by the following equivalence rules, axiom rule and congruence rule for operators:

$$\mathsf{Ref}\ \frac{}{t \approx_E t}\ t \in T_\Sigma V \qquad \mathsf{Sym}\ \frac{t \approx_E t'}{t' \approx_E t} \qquad \mathsf{Trans}\ \frac{t \approx_E t' \quad t' \approx_E t''}{t \approx_E t''}$$

$$\mathsf{Axiom}\ \frac{}{t\{s_1/x_1, \ldots, s_n/x_n\} \ \approx_E \ t'\{s_1/x_1, \ldots, s_n/x_n\}}\ \begin{array}{l}(x_1, .., x_n \vdash t \equiv t') \in E\,, \\ s_1, \ldots, s_n \in T_\Sigma V\end{array}$$

$$\mathsf{Cong}\ \frac{t_1 \approx_E t'_1\,,\ \ldots\ ,\ t_k \approx_E t'_k}{\mathsf{o}(t_1, \ldots, t_k)\ \approx_E\ \mathsf{o}(t'_1, \ldots, t'_k)}\ \mathsf{o} \in \Sigma \text{ with } |\mathsf{o}| = k$$

where $t\{s_1/x_1, \ldots, s_n/x_n\}$ denotes the term obtained by simultaneously substituting the terms $s_1, \ldots, s_n$ for the variables $x_1, .., x_n$ in the term t. The interpretation of each operator on the carrier set $T_\Sigma(V)/_{\approx_E}$ is given syntactically:

$$[\![\mathsf{o}]\!]([t_1]_{\approx_E}, \ldots, [t_k]_{\approx_E}) = [\mathsf{o}(t_1, \ldots, t_k)]_{\approx_E}\,.$$

This construction gives rise to a left adjoint to the forgetful functor $U_{\mathbb{T}} : \mathbb{T}\text{-}\mathbf{Alg} \to \mathbf{Set}$ and thus to an associated monad $\mathbf{T}_{\mathbb{T}}$ on **Set**.

Moreover, the following properties are known to hold.

1. The adjunction is monadic; *i.e.*, the category $\mathbb{T}$-**Alg** of algebras for the theory $\mathbb{T}$ is isomorphic to the category of Eilenberg-Moore algebras for the monad $\mathbf{T}_{\mathbb{T}}$.

2. The category $\mathbb{T}$-**Alg** is cocomplete.

3. The monad $\mathbf{T}_{\mathbb{T}}$ is finitary; *i.e.*, it preserves filtered colimits.

Example 3.1.1 (continued)**.** The free algebra on a set V for the algebraic theory $\mathbb{G}$ of groups is the free group generated by the set V in the usual sense (see *e.g.* [Hungerford 1996]). It follows that the category of groups is cocomplete and monadic over **Set**, and that the free group monad on **Set** is finitary.

3.2 Free constructions for inductive equational systems

We present a categorical construction of free algebras for inductive equational systems. The construction of free algebras for an equational system $\mathbb{S}$ is that of a left adjoint to the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$. Since the forgetful functor decomposes as $\mathbb{S}\text{-}\mathbf{Alg} \overset{J_{\mathbb{S}}}{\hookrightarrow} \Sigma\text{-}\mathbf{Alg} \xrightarrow{U_\Sigma} \mathscr{C}$ for Σ the functorial signature of $\mathbb{S}$, we construct a left adjoint to $U_{\mathbb{S}}$ in two stages, as the composition of a left adjoint to U_Σ followed by a left adjoint to $J_{\mathbb{S}}$. A construction for the former has already been studied in the literature (see *e.g.* [Adámek 1974, Lehmann and Smyth 1981, Smyth and Plotkin 1982, Barr and Wells 1985, Adámek and Trnková 1990]) and we briefly review it in Section 3.2.2. Thus, in Section 3.2.3, we concentrate on obtaining a reflection to the embedding of $\mathbb{S}$-**Alg** into Σ-**Alg**. The construction to be developed depends on the key concepts of *algebra cospan* and *algebraic coequalizer*, which are introduced in Section 3.2.1. In Section 3.2.4, we further discuss the construction of free algebras for inductive monadic equational systems.

Overall, by means of Theorems 3.2.6 and 3.2.8, we establish the following theorem.

Theorem 3.2.1. *For an inductive equational system $\mathbb{S}$ on a category $\mathscr{C}$, the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint.*

3.2.1 Algebra cospans and algebraic coequalizers

We introduce the notions of Σ-algebra cospan and of Σ-algebraic coequalizer for an endofunctor Σ on a category $\mathscr{C}$. Also, for $\mathscr{C}$ cocomplete and Σ ω-cocontinuous, we provide a construction for reflecting Σ-algebra cospans into Σ-algebras, which in turn yields a construction for Σ-algebraic coequalizers.

Definition 3.2.2 (Algebra cospan)**.** For an endofunctor Σ on a category $\mathscr{C}$, a Σ-*algebra cospan* is a cospan of the form $(Z \to Z_1 \leftarrow \Sigma Z)$. A homomorphism (h, h_1) from a Σ-algebra cospan $(Z \xrightarrow{c} Z_1 \xleftarrow{t} \Sigma Z)$ to another one $(Z' \xrightarrow{c'} Z_1' \xleftarrow{t'} \Sigma Z')$ is given by a pair of morphisms $(h : Z \to Z',\ h_1 : Z_1 \to Z_1')$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \Sigma Z \arrow[d, "t"'] \arrow[rrd, "\Sigma h"] & & \\
Z \arrow[r, "c"] \arrow[rrd, "h"'] & Z_1 \arrow[rrd, "h_1"] & & \Sigma Z' \arrow[d, "t'"] \\
 & & Z' \arrow[r, "c'"'] & Z_1'
\end{tikzcd}$$

Σ-algebra cospans and homomorphisms form the category Σ-**AlgCoSp**.

We will henceforth regard Σ-**Alg** as a full subcategory of Σ-**AlgCoSp** via the embedding that maps $(Z, t : \Sigma Z \to Z)$ to $(Z \xrightarrow{\mathrm{id}} Z \xleftarrow{t} \Sigma Z)$.

Definition 3.2.3 (Algebraic coequalizer)**.** Let Σ be an endofunctor on a category $\mathscr{C}$. By a Σ-*algebraic coequalizer* of a parallel pair $l, r : Y \to Z$ into a Σ-algebra (Z, t) we mean a universal Σ-algebra homomorphism z from (Z, t) coequalizing the parallel pair.

$$\begin{tikzcd}
 & \Sigma Z \arrow[r, "\Sigma z"] \arrow[rrd, "\Sigma h"'] \arrow[dd, "t"] & \Sigma Z' \arrow[rd, dashed, "\Sigma h'"] \arrow[dd, "t'"] & \\
 & & & \Sigma W \arrow[dd, "u"] \\
Y \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & Z \arrow[r, "z"] \arrow[rrd, "h"'] & Z' \arrow[rd, dashed, "\exists!\, h'"] & \\
 & & & W
\end{tikzcd}$$

We present a construction for reflecting Σ-algebra cospans into Σ-algebras, *i.e.*, a construction of free Σ-algebras on Σ-algebra cospans.

Theorem 3.2.4. *Let Σ be an endofunctor on a category $\mathscr{C}$. If $\mathscr{C}$ is cocomplete and Σ is ω-cocontinuous then* Σ-**Alg** *is a full reflective subcategory of* Σ-**AlgCoSp**.

Proof. Given a Σ-algebra cospan $(c_0 : Z_0 \to Z_1 \leftarrow \Sigma Z_0 : t_0)$ we inductively construct a Σ-algebra $t_\infty : \Sigma Z_\infty \to Z_\infty$ as follows:

$$\begin{tikzcd}
\Sigma Z_0 \arrow[r, "\Sigma c_0"] \arrow[rd, "t_0"] & \Sigma Z_1 \arrow[r, "\Sigma c_1"] \arrow[rd, "t_1"] \arrow[d, phantom, "\mathsf{po}"] & \Sigma Z_2 \arrow[r, "\Sigma c_2"] \arrow[rd, "t_2"] \arrow[d, phantom, "\mathsf{po}"] & \Sigma Z_3 & \cdots & \Sigma Z_\infty \arrow[d, dashed, "\exists!\, t_\infty"] \\
Z_0 \arrow[r, "c_0"'] & Z_1 \arrow[r, "c_1"'] & Z_2 \arrow[r, "c_2"'] & Z_3 & \cdots & Z_\infty \ \mathsf{colim}
\end{tikzcd} \tag{3.1}$$

where

- $Z_{n+1} \xrightarrow{c_{n+1}} Z_{n+2} \xleftarrow{t_{n+1}} \Sigma Z_{n+1}$ is a pushout of $Z_{n+1} \xleftarrow{t_n} \Sigma Z_n \xrightarrow{\Sigma c_n} \Sigma Z_{n+1}$, for all $n \geq 0$;
- Z_∞ with $\{\, \overline{c}_n : Z_n \to Z_\infty \,\}_{n \geq 0}$ is a colimit of the ω-chain $\{\, c_n \,\}_{n \geq 0}$; and

- t_∞ is the mediating map from the colimiting cone $\{\, \Sigma\overline{c}_n : \Sigma Z_n \to \Sigma Z_\infty \,\}_{n\geq 0}$ to the cone $\{\, \overline{c}_{n+1} \circ t_n \,\}_{n\geq 0}$ of the ω-chain $\{\, \Sigma c_n \,\}_{n\geq 0}$.

We now show that the map $(\overline{c}_0, \overline{c}_1) : (Z_0 \to Z_1 \leftarrow \Sigma Z_0) \longrightarrow (Z_\infty \xrightarrow{\mathrm{id}} Z_\infty \leftarrow \Sigma Z_\infty)$ in Σ-**AlgCoSp** is universal. For this, consider another map

$$(h_0, h_1) : (Z_0 \to Z_1 \leftarrow \Sigma Z_0) \longrightarrow (W \xrightarrow{\mathrm{id}} W \xleftarrow{u} \Sigma W)$$

and perform the following construction

$$\begin{tikzcd}
\Sigma Z_0 \arrow[r, "\Sigma c_0"] \arrow[dr, "t_0"] \arrow[drrrrr, "\Sigma h_0"] & \Sigma Z_1 \arrow[r, "\Sigma c_1"] \arrow[dr, "t_1"] \arrow[drrrr, "\Sigma h_1"] & \Sigma Z_2 \arrow[drrr, "\Sigma h_2"] & \cdots & \Sigma Z_\infty \arrow[d, "t_\infty"] \arrow[dr, "\Sigma h_\infty"] & \\
Z_0 \arrow[r, "c_0"] \arrow[drrrrr, "h_0"] & Z_1 \arrow[r, "c_1"] \arrow[drrrr, "h_1"] & Z_2 \arrow[drrr, "h_2"] & \cdots & Z_\infty \arrow[dr, "h_\infty"] & \Sigma W \arrow[d, "u"] \\
 & & & & & W
\end{tikzcd}$$

where

- for $n \geq 0$, h_{n+2} is the mediating map from the pushout Z_{n+2} to W with respect to the cone $(h_{n+1} : Z_{n+1} \to W \leftarrow \Sigma Z_{n+1} : u \circ \Sigma h_{n+1})$; and
- h_∞ is the mediating map from the colimit Z_∞ to W with respect to the cone $\{\, h_n \,\}_{n\geq 0}$ of the ω-chain $\{\, c_n \,\}_{n\geq 0}$.

As, for all $n \geq 0$, $u \circ \Sigma h_\infty \circ \Sigma\overline{c}_n = h_\infty \circ t_\infty \circ \Sigma\overline{c}_n$, it follows that h_∞ is a Σ-algebra homomorphism. Hence, (h_0, h_1) factors as $(h_\infty, h_\infty) \circ (\overline{c}_0, \overline{c}_1)$ in Σ-**AlgCoSp**.

We finally establish the uniqueness of such factorizations. Indeed, for any homomorphism $h : (Z_\infty, t_\infty) \to (W, u)$ such that $h \circ \overline{c}_1 = h_1$, it follows by induction that $h \circ \overline{c}_n = h_n$ for all $n \geq 0$, and hence that $h = h_\infty$. $\square$

A construction for Σ-algebraic coequalizers follows as a corollary.

Corollary 3.2.5. *Let Σ be an endofunctor on a category $\mathscr{C}$. If $\mathscr{C}$ is cocomplete and Σ is ω-cocontinuous then Σ-algebraic coequalizers exist. If, in addition, Σ is epicontinuous then Σ-algebraic coequalizers are epimorphic in $\mathscr{C}$.*

Proof. Consider the following construction:

$$\begin{tikzcd}
 & \Sigma Z \arrow[rr, "\Sigma z"] \arrow[d, "t"'] \arrow[dr, "c \circ t"] & & \Sigma Z' \arrow[d, "t'"] \\
Y \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & Z \arrow[r, two heads, "c", "\mathsf{coeq}"'] & Z_1 \arrow[r, "z_1"] \arrow[ur, phantom, "\mathsf{reflect}"] & Z'
\end{tikzcd} \tag{3.2}$$

Given a Σ-algebra (Z, t) and a parallel pair $l, r : Y \to Z$, let $c : Z \twoheadrightarrow Z_1$ be a coequalizer of the pair l, r in $\mathscr{C}$. By Theorem 3.2.4, we can construct a reflection (z, z_1) :

$(Z \xrightarrow{c} Z_1 \xleftarrow{c \circ t} \Sigma Z) \longrightarrow (Z' \xrightarrow{\mathrm{id}} Z' \xleftarrow{t'} \Sigma Z')$ for the Σ-algebra cospan $(Z \xrightarrow{c} Z_1 \xleftarrow{c \circ t} \Sigma Z)$. It follows that the homomorphism $z = z_1 \circ c : (Z, t) \to (Z', t')$ is a Σ-algebraic coequalizer of the pair l, r.

Now recall that the map $z : Z \to Z'$ is given as $\overline{c}_0 : (Z_0, t) \to (Z_\infty, t_\infty)$ in the construction (3.1) where c_0 and t_0 are respectively taken to be the coequalizer c of l, r and the composite $c \circ t$. If Σ is epicontinuous, the ω-chain $\{ c_n : Z_n \to Z_{n+1} \}_{n \geq 0}$ in (3.1) consists of epimorphisms, and hence this is also the case for its colimiting cone $\{ \overline{c}_n : Z_n \to Z_\infty \}_{n \geq 0}$. □

3.2.2 Construction of free algebras for endofunctors

The following result describes a well-known condition for the existence of free algebras for endofunctors (see *e.g.* [Adámek 1974]).

Theorem 3.2.6. *Let Σ be an endofunctor on a category $\mathscr{C}$. If $\mathscr{C}$ is cocomplete and Σ is ω-cocontinuous, then the forgetful functor $U_\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint.*

Proof. Let X be an object in $\mathscr{C}$. As the endofunctor $X + \Sigma(-)$ on $\mathscr{C}$ is ω-cocontinuous, by Theorem 3.2.4 a free $(X + \Sigma(-))$-algebra $(TX, [\eta_X, \tau_X])$ on the initial $(X + \Sigma(-))$-algebra cospan $(0 \xrightarrow{!} X + \Sigma 0 \xleftarrow{\mathrm{id}} X + \Sigma 0)$ is constructed as follows:

$$\begin{tikzcd}
 & X + \Sigma 0 \arrow[r, "X+\Sigma !"] \arrow[dr, "\mathrm{id}"] \arrow[dr, phantom, "\mathsf{po}", near start, shift left=6] & X + \Sigma(X + \Sigma 0) \arrow[dr, "\mathrm{id}"] & \cdots\cdots & X + \Sigma(TX) \arrow[d, dashed, "\cong \,\exists! [\eta_X, \tau_X]"] \\
0 \arrow[rr, "!"] & & X + \Sigma 0 \arrow[r, "X+\Sigma !"] & X + \Sigma(X + \Sigma 0) \quad \cdots\cdots & TX \ \mathsf{colim}
\end{tikzcd} \tag{3.3}$$

where 0 is an initial object and $!$ is the unique map. As $(TX, [\eta_X, \tau_X])$ is an initial $(X + \Sigma(-))$-algebra, it follows that $(TX, \tau_X : \Sigma TX \to TX)$ is a free Σ-algebra on X with universal map $\eta_X : X \to TX$. □

In the construction above, the carrier object TX is given as a colimit of the ω-chain $\{ f_n : X_n \to X_{n+1} \}_{n \geq 0}$, inductively defined by setting $X_0 = 0$, $f_0 = \,!$ and, for $n \geq 0$, $X_{n+1} = X + \Sigma X_n$, $f_{n+1} = X + \Sigma f_n$. The intuition behind the construction of TX, in which Σ represents a signature and X an object of variables, is that each object X_n consists of terms of depth at most n built up from operators in Σ and variables in X. The object TX is intuitively the union of the sequence of objects $\{ X_n \}_{n \geq 0}$, *i.e.*, it intuitively consists of terms of finite depth.

Example 3.2.7 (continued)**.** Recall that the functorial signature $F_{\Sigma_\mathbb{G}}$ for the equational system of groups is given by $F_{\Sigma_\mathbb{G}} X = 1 + X + X^2$. We consider the above construction for $F_{\Sigma_\mathbb{G}}$ to obtain free $F_{\Sigma_\mathbb{G}}$-algebras. For a set V of variables, V_0 is defined to be the empty set and the sets V_n, for $n \geq 1$, are inductively defined as

$$\begin{array}{rcl} V_n & = & V + F_{\Sigma_\mathbb{G}}(V_{n-1}) \;=\; V + 1 + V_{n-1} + {V_{n-1}}^2 \\ & \cong & \{ v \mid v \in V \} \uplus \{ \mathsf{e} \} \uplus \{ \mathsf{i}(t) \mid t \in V_{n-1} \} \uplus \{ \mathsf{m}(t, t') \mid t, t' \in V_{n-1} \}\,. \end{array}$$

The free $F_{\Sigma_{\mathbb{G}}}$-algebra (TV, τ_V) on V has as carrier the set $TV = \bigcup_{n\geq 0} V_n$, which is inductively given by the following grammar

$$t \in TV \ ::= \ v \ | \ \mathsf{e} \ | \ \mathsf{i}(t) \ | \ \mathsf{m}(t,t') \qquad \text{for } v \in V,\ t,t' \in TV\,.$$

The algebra structure $\tau_V = [\llbracket \mathsf{e} \rrbracket, \llbracket \mathsf{i} \rrbracket, \llbracket \mathsf{m} \rrbracket] : 1 + TV + TV^2 \to TV$ is given by $\llbracket \mathsf{e} \rrbracket() = \mathsf{e}$, $\llbracket \mathsf{i} \rrbracket(t) = \mathsf{i}(t)$, $\llbracket \mathsf{m} \rrbracket(t,t') = \mathsf{m}(t,t')$.

3.2.3 Construction of free algebras for equational systems

We present a construction of a left adjoint to the embedding $\mathbb{S}$-**Alg** $\hookrightarrow$ Σ-**Alg** for an inductive equational system $\mathbb{S}$ with functorial signature Σ; that is, a construction of free $\mathbb{S}$-algebras over Σ-algebras.

Theorem 3.2.8. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an inductive equational system. Then the embedding $\mathbb{S}$-**Alg** $\hookrightarrow$ Σ-**Alg** has a left adjoint. Furthermore, the universal homomorphisms from Σ-algebras to their free $\mathbb{S}$-algebras are epimorphic in $\mathscr{C}$.*

Proof. By Corollary 3.2.5, Σ-algebraic coequalizers exist and they are epimorphic in $\mathscr{C}$. We claim that for any Σ-algebra (X,s), a Σ-algebraic coequalizer, say $q : (X,s) \to (\widetilde{X}, \widetilde{s})$, of the pair $L(X,s)^\diamond, R(X,s)^\diamond : \Gamma X \to X$ into the Σ-algebra (X,s) is a free $\mathbb{S}$-algebra on the Σ-algebra (X,s). See the diagram (3.4) below.

First, we show that $(\widetilde{X}, \widetilde{s})$ is an $\mathbb{S}$-algebra. Since the map q coequalizes the pair $L(X,s)^\diamond, R(X,s)^\diamond$ and, being a Σ-algebra homomorphism $(X,s) \to (\widetilde{X}, \widetilde{s})$, also yields Γ-algebra homomorphisms $q : L(X,s) \to L(\widetilde{X}, \widetilde{s})$ and $q : R(X,s) \to R(\widetilde{X}, \widetilde{s})$, it follows that $L(\widetilde{X}, \widetilde{s})^\diamond \circ \Gamma q = R(\widetilde{X}, \widetilde{s})^\diamond \circ \Gamma q$. Since q is epimorphic in $\mathscr{C}$ and Γ is epicontinuous, the map Γq is epimorphic and thus we have $L(\widetilde{X}, \widetilde{s})^\diamond = R(\widetilde{X}, \widetilde{s})^\diamond$.

Second, for every homomorphism h from (X,s) to an $\mathbb{S}$-algebra (Y,t), we show that the homomorphism h uniquely factors through $q : (X,s) \to (\widetilde{X}, \widetilde{s})$. As q is an algebraic coequalizer of $L(X,s)^\diamond, R(X,s)^\diamond$, it is enough to show that the map h coequalizes $L(X,s)^\diamond, R(X,s)^\diamond$. This directly follows from the fact that (Y,t) is an $\mathbb{S}$-algebra because $L(Y,t)^\diamond = R(Y,t)^\diamond$ implies that Γh equalizes $L(Y,t)^\diamond, R(Y,t)^\diamond$, which in turn entails that h coequalizes $L(X,s)^\diamond, R(X,s)^\diamond$. $\square$

By the construction (3.2) depending on the construction (3.1), the free $\mathbb{S}$-algebra $(\widetilde{X}, \widetilde{s})$ on a Σ-algebra (X,s) with universal homomorphism $q : (X,s) \to (\widetilde{X}, \widetilde{s})$ is constructed as follows:

$$\begin{tikzcd}
 & \Sigma X \arrow[r, two heads, "\Sigma q_0"] \arrow[d, "s"] \arrow[dr, "s_0"] & \Sigma X_1 \arrow[r, two heads, "\Sigma q_1"] \arrow[dr, "s_1"] \arrow[d, phantom, "\mathsf{po}"] & \Sigma X_2 \arrow[r, two heads, "\Sigma q_2"] \arrow[dr, "s_2"] \arrow[d, phantom, "\mathsf{po}"] & \Sigma X_3 & \cdots\cdots & \Sigma\widetilde{X} \arrow[d, dashed, "\exists!\,\widetilde{s}"] \\
\Gamma X \arrow[r, shift left, "L(X,s)^\diamond"] \arrow[r, shift right, "R(X,s)^\diamond"'] & X \arrow[r, two heads, "q_0", "\mathsf{coeq}"'] \arrow[rrrrr, bend right, two heads, "q=\overline{q}_0"'] & X_1 \arrow[r, two heads, "q_1"] \arrow[rrrr, bend right, two heads, "\overline{q}_1"'] & X_2 \arrow[r, two heads, "q_2"] & X_3 & \cdots\cdots & \widetilde{X}_{\ \mathsf{colim}}
\end{tikzcd} \tag{3.4}$$

Here, the map q_0 is a coequalizer of the parallel pair $L(X,s)^\diamond, R(X,s)^\diamond$. The map s_0 is set to be $q_0 \circ s$, and the maps q_i and s_i, for $i \geq 1$, are inductively defined by letting X_{i+1} with q_i and s_i be a pushout of s_{i-1} and $\Sigma(q_{i-1})$. The carrier object $\widetilde{X}$ with maps $\{\, \overline{q}_i : X_i \to \widetilde{X} \,\}_{i\geq 0}$ is a colimit of the ω-chain $\{\, q_i \,\}_{i\geq 0}$ and the structure map $\widetilde{s}$ is the unique mediating map from the colimit $\Sigma\widetilde{X}$ of the ω-chain $\{\, \Sigma q_i \,\}_{i\geq 0}$ to the cone $\{\, \overline{q}_{i+1} \circ s_i \,\}_{i\geq 0}$.

The intuition behind the construction of X_1 is that of quotienting X according to the equation $L \equiv R$. For $n > 1$, the construction of X_n from X_{n-1} as a pushout is intuitively quotienting the object X_{n-1} by congruence rules for the operators. Therefore, the intuition behind the construction of $\widetilde{X}$ is that of quotienting the object X by the equation $L \equiv R$ and congruence rules.

Example 3.2.9 (continued)**.** For the equational system of groups

$$\begin{array}{rlcccccccc}
\mathbb{S}_{\mathbb{G}} = (& \mathbf{Set} : F_{\Sigma_{\mathbb{G}}} \\
& \rhd & [\![\{x\}]\!] & + & [\![\{x\}]\!] & + & [\![\{x\}]\!] & + & [\![\{x,y,z\}]\!] \\
& \vdash & [\ [\![\mathsf{m}(x,\mathsf{e})]\!] & , & [\![\mathsf{m}(x,\mathsf{i}(x))]\!] & , & [\![\mathsf{m}(\mathsf{i}(x),x)]\!] & , & [\![\mathsf{m}(\mathsf{m}(x,y),z)]\!]\ ] \\
& \equiv & [\quad [\![x]\!] & , & [\![\mathsf{e}]\!] & , & [\![\mathsf{e}]\!] & , & [\![\mathsf{m}(x,\mathsf{m}(y,z))]\!]\ ]\ ) ,
\end{array}$$

we consider the construction of the free $\mathbb{S}_{\mathbb{G}}$-algebra $(\widetilde{TV}, \widetilde{\tau_V})$ over the free $F_{\Sigma_{\mathbb{G}}}$-algebra (TV, τ_V) constructed in Example 3.2.7.

As the map $q_0 : TV \twoheadrightarrow (TV)_1$ is the universal map in **Set** that coequalizes the pairs

$$\begin{array}{rcrcccc}
[\![\mathsf{m}(x,\mathsf{e})]\!](TV,\tau_V) & , & [\![x]\!](TV,\tau_V) & : & TV & \longrightarrow & TV \\
[\![\mathsf{m}(x,\mathsf{i}(x))]\!](TV,\tau_V) & , & [\![\mathsf{e}]\!](TV,\tau_V) & : & TV & \longrightarrow & TV \\
[\![\mathsf{m}(\mathsf{i}(x),x)]\!](TV,\tau_V) & , & [\![\mathsf{e}]\!](TV,\tau_V) & : & TV & \longrightarrow & TV \\
[\![\mathsf{m}(\mathsf{m}(x,y),z)]\!](TV,\tau_V) & , & [\![\mathsf{m}(x,\mathsf{m}(y,z))]\!](TV,\tau_V) & : & (TV)^3 & \longrightarrow & TV
\end{array}$$

it follows from the standard construction of coequalizers in **Set** that the set $(TV)_1$ is the quotient set of TV under the equivalence relation $\approx_1$ generated by the following rules:

$$\frac{t \in TV}{\mathsf{m}(t,\mathsf{e}) \approx_1 t} \qquad \frac{t \in TV}{\mathsf{m}(t,\mathsf{i}(t)) \approx_1 \mathsf{e}} \qquad \frac{t \in TV}{\mathsf{m}(\mathsf{i}(t),t) \approx_1 \mathsf{e}} \qquad \frac{t,t',t'' \in TV}{\mathsf{m}(\mathsf{m}(t,t'),t'') \approx_1 \mathsf{m}(t,\mathsf{m}(t',t''))}$$

The map q_0 sends a term t to the equivalence class $[t]_{\approx_1}$.

We observe that a pushout of a surjective map $e : A \twoheadrightarrow B$ and a map $f : A \to C$ in **Set** is given by the quotient set $C/_\sim$ of the set C under the equivalence relation $\sim$ generated by the rule $f(a) \sim f(a')$ in C for all $a, a' \in A$ such that $e(a) = e(a')$ in B; with the surjective map $e' : C \twoheadrightarrow C/_\sim$ sending an element c to its equivalence class $[c]_\sim$, and the map $f' : B \to C/_\sim$ sending an element b to $e'(f(\widetilde{b}))$ for $\widetilde{b}$ an element of A such that $e(\widetilde{b}) = b$. From this observation, we have that the sets $(TV)_n$, for $n > 1$, are respectively the quotient sets of TV under the equivalence relations $\approx_n$ inductively generated by the following rules:

$$\frac{t \approx_{n-1} s}{t \approx_n s} \qquad \frac{}{\mathsf{e} \approx_n \mathsf{e}} \qquad \frac{t \approx_{n-1} s}{\mathsf{i}(t) \approx_n \mathsf{i}(s)} \qquad \frac{t \approx_{n-1} s \quad t' \approx_{n-1} s'}{\mathsf{m}(t,t') \approx_n \mathsf{m}(s,s')}$$

The map q_{n-1} sends $[t]_{\approx_{n-1}}$ to $[t]_{\approx_n}$.

Thus the object $\widetilde{TV}$, being the colimit of the ω-chain $\{\,(TV)_n\,\}_{n\geq 1}$ in **Set**, is given as the quotient set of TV under the equivalence relation $\approx$ given by the following rules:

$$\mathsf{Ref}\,\frac{t \in TV}{t \approx t} \qquad \mathsf{Sym}\,\frac{t \approx s}{s \approx t} \qquad \mathsf{Trans}\,\frac{t \approx s \quad s \approx r}{t \approx r}$$

$$\mathsf{Axiom_1}\,\frac{t \in TV}{\mathsf{m}(t,\mathsf{e}) \approx t} \quad \mathsf{Axiom_2}\,\frac{t \in TV}{\mathsf{m}(t,\mathsf{i}(t)) \approx \mathsf{e}} \quad \mathsf{Axiom_3}\,\frac{t \in TV}{\mathsf{m}(\mathsf{i}(t),t) \approx \mathsf{e}}$$

$$\mathsf{Axiom_4}\,\frac{t, t', t'' \in TV}{\mathsf{m}(\mathsf{m}(t,t'),t'') \approx \mathsf{m}(t,\mathsf{m}(t',t''))}$$

$$\mathsf{Cong\text{-}e}\,\frac{}{\mathsf{e} \approx \mathsf{e}} \qquad \mathsf{Cong\text{-}i}\,\frac{t \approx s}{\mathsf{i}(t) \approx \mathsf{i}(s)} \qquad \mathsf{Cong\text{-}m}\,\frac{t \approx s \quad t' \approx s'}{\mathsf{m}(t,t') \approx \mathsf{m}(s,s')}$$

The map q sends a term t to the equivalence class $[t]_{\approx}$.

3.2.4 Construction of free algebras for monadic equational systems

One can construct free algebras for inductive monadic equational systems via the encoding of these systems into inductive (ordinary) equational systems given in Section 2.3. In this section, we simplify the construction (see Corollary 3.2.12).

Definition 3.2.10. A monadic equational system $\mathbb{S} = (\mathscr{C} : \mathbf{T} \rhd \Gamma \vdash L \equiv R)$ is called *inductive* if the category $\mathscr{C}$ is cocomplete, and the underlying endofunctor of the monad $\mathbf{T}$ and the functorial context Γ are epicontinuous and ω-cocontinuous.

Theorem 3.2.11. *Let $\mathbb{S} = (\mathscr{C} : \mathbf{T} \rhd \Gamma \vdash L \equiv R)$ be an inductive monadic equational system with $\mathbf{T} = (T, \eta, \mu)$. For an Eilenberg-Moore algebra (X, s) of the monad $\mathbf{T}$, every T-algebraic coequalizer of $L(X,s)^\diamond$, $R(X,s)^\diamond$ into (X,s) yields a free $\mathbb{S}$-algebra $(\widetilde{X}, \widetilde{s})$ over (X,s). Hence, this construction provides a left adjoint to $J : \mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow \mathscr{C}^{\mathbf{T}}$.*

$$\begin{tikzcd}
 & TX \arrow[r, two heads, "Tq"] \arrow[d, "s"] & T\widetilde{X} \arrow[d, "\widetilde{s}"] \\
\Gamma X \arrow[r, shift left, "L(X,s)^\diamond"] \arrow[r, shift right, "R(X,s)^\diamond"'] & X \arrow[r, two heads, "q", "\mathsf{algcoeq}"'] & \widetilde{X}
\end{tikzcd}$$

Proof. Recall from Section 2.3 that we have the equivalent equational system

$$\overline{\mathbb{S}} = \big(\mathscr{C} : T \rhd (\Gamma_{\mathbf{T}} + \Gamma) \vdash [L_{\mathbf{T}}, \overline{L}] \equiv [R_{\mathbf{T}}, \overline{R}]\big)$$

and the following situation

$$\begin{tikzcd}
 & & \Gamma\text{-}\mathbf{Alg} & \Gamma_{\mathbf{T}}\text{-}\mathbf{Alg} \\
\mathbb{S}\text{-}\mathbf{Alg} = \overline{\mathbb{S}}\text{-}\mathbf{Alg} \arrow[r, hook, "J"'] & \mathscr{C}^{\mathbf{T}} \arrow[ur, shift left, "L"] \arrow[ur, shift right, "R"'] \arrow[r, hook, "J_{\mathbf{T}}"'] & T\text{-}\mathbf{Alg} \arrow[u, shift left, "\overline{R}"'] \arrow[u, shift right, "\overline{L}"] \arrow[ur, shift left, "L_{\mathbf{T}}"] \arrow[ur, shift right, "R_{\mathbf{T}}"']
\end{tikzcd}$$

As the equational system $\overline{\mathbb{S}}$ is also inductive, the embedding $J_{\mathbf{T}}\,J$ has a left adjoint $\overline{K} : T\text{-}\mathbf{Alg} \to \overline{\mathbb{S}}\text{-}\mathbf{Alg}$ constructed by means of T-algebraic coequalizers (see Theorem 3.2.8). It follows that the composite functor $\overline{K}\,J_{\mathbf{T}} : \mathscr{C}^{\mathbf{T}} \to \mathbb{S}\text{-}\mathbf{Alg}$ is a left adjoint to the embedding J.

From the construction of the reflection $\overline{K} : T\text{-}\mathbf{Alg} \to \overline{\mathbb{S}}\text{-}\mathbf{Alg}$ given in Theorem 3.2.8, we see that the $\overline{\mathbb{S}}$-algebra $\overline{K}(J_{\mathbf{T}}(X,s))$ for $(X,s) \in \mathscr{C}^{\mathbf{T}}$ is given by a T-algebraic coequalizer of the pair $[L_{\mathbf{T}}(J_{\mathbf{T}}(X,s))^{\diamond}, \overline{L}(J_{\mathbf{T}}(X,s))^{\diamond}]$ and $[R_{\mathbf{T}}(J_{\mathbf{T}}(X,s))^{\diamond}, \overline{R}(J_{\mathbf{T}}(X,s))^{\diamond}]$ into $J_{\mathbf{T}}(X,s)$, which is a T-algebraic coequalizer of the pair $L(X,s)^{\diamond}$, $R(X,s)^{\diamond}$ into $J_{\mathbf{T}}(X,s)$ because $L_{\mathbf{T}}(J_{\mathbf{T}}(X,s))^{\diamond} = R_{\mathbf{T}}(J_{\mathbf{T}}(X,s))^{\diamond}$ and $\overline{L}\,J_{\mathbf{T}} = L$, $\overline{R}\,J_{\mathbf{T}} = R$. $\square$

For each object X in $\mathscr{C}$, as $(TX, \mu_X : TTX \to TX)$ is a free Eilenberg-Moore algebra on X, we have the following corollary.

Corollary 3.2.12. *Let* $\mathbb{S} = (\mathscr{C} : \mathbf{T} \rhd \Gamma \vdash L \equiv R)$ *be an inductive monadic equational system with* $\mathbf{T} = (T, \eta, \mu)$*. For each object* X *in* $\mathscr{C}$*, every* T*-algebraic coequalizer of* $L(TX, \mu_X)^{\diamond}$*,* $R(TX, \mu_X)^{\diamond}$ *into* (TX, μ_X) *yields a free* $\mathbb{S}$*-algebra on* X*.*

3.3 Properties of inductive equational systems

For inductive equational systems, we show

1. the monadicity and cocompleteness of categories of algebras (see Section 3.3.1);

2. the epicontinuity and ω-cocontinuity of representing monads (see Section 3.3.2); and

3. the cocompleteness of categories of inductive equational systems (see Section 3.3.3).

Overall, we establish the following theorem (by means of Theorem 3.3.6, Theorem 3.3.9 and Corollary 3.3.8).

Theorem 3.3.1. *For an inductive equational system* $\mathbb{S}$ *on a category* $\mathscr{C}$*, the category* $\mathbb{S}$-$\mathbf{Alg}$ *is cocomplete, the forgetful functor* $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ *is monadic, and the representing monad of* $\mathbb{S}$ *is epicontinuous and* ω*-cocontinuous.*

3.3.1 Properties of categories of algebras

We first show some general properties of categories of algebras for equational systems, and then show that categories of algebras for inductive equational systems are cocomplete and monadic over their base categories.

For an equational system $\mathbb{S}$ with functorial signature Σ, the category $\mathbb{S}$-$\mathbf{Alg}$ is a *replete*, also called *isomorphism-closed*, subcategory of Σ-$\mathbf{Alg}$ in the following sense.

Proposition 3.3.2. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system. For any isomorphic Σ-algebras $(X, s) \cong (Y, t)$, the following holds:*

$$(X, s) \in \mathbb{S}\text{-}\mathbf{Alg} \quad \Longrightarrow \quad (Y, t) \in \mathbb{S}\text{-}\mathbf{Alg}\,.$$

Proof. For an isomorphism $p : (X, s) \cong (Y, t) : q$, we have that $L(Y, t)^\diamond = p \circ L(X, s)^\diamond \circ \Gamma(q)$ and $R(Y, t)^\diamond = p \circ R(X, s)^\diamond \circ \Gamma(q)$. Thus, $L(X, s) = R(X, s)$ implies $L(Y, t) = R(Y, t)$. ☐

We recall two well-known properties for endofunctors Σ on a category $\mathscr{C}$.

- $(X, s : \Sigma X \to X)$ is a free Σ-algebra on $A \in \mathscr{C}$ with unit map $a : A \to X$ if and only if $(X, [a, s] : A + \Sigma X \to X)$ is an initial $(A + \Sigma(-))$-algebra.
- If the forgetful functor Σ-$\mathbf{Alg} \to \mathscr{C}$ has a left adjoint then it is monadic.

These results extend to equational systems.

Proposition 3.3.3. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system. For any object $A \in \mathscr{C}$, let $\mathbb{S}^A$ be the equational system given by $(\mathscr{C} : (A + \Sigma(-)) \rhd \Gamma \vdash L\,U_A \equiv R\,U_A)$ where U_A denotes the forgetful functor $(A + \Sigma(-))$-$\mathbf{Alg} \to \Sigma$-$\mathbf{Alg}$. Then, it holds that $(X, s : \Sigma X \to X)$ is a free $\mathbb{S}$-algebra on $A \in \mathscr{C}$ with unit map $a : A \to X$ if and only if $(X, [a, s] : A + \Sigma X \to X)$ is an initial $\mathbb{S}^A$-algebra.*

Proof. By definition of the system $\mathbb{S}^A$, an $\mathbb{S}^A$-algebra $(X, [a, s])$ is simply given by a pair consisting of an $\mathbb{S}$-algebra (X, s) and a map $a : A \to X$. The conclusion of the proposition trivially follows from this observation. ☐

Proposition 3.3.4. *Let $\mathbb{S}$ be an equational system on a category $\mathscr{C}$. If the forgetful functor $U_{\mathbb{S}} : \mathbb{S}$-$\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, then it is monadic.*

Proof. To show the monadicity of $U_{\mathbb{S}}$, by Beck's theorem [Mac Lane 1998, Theorem 1 of Section VI.7], it is enough to show that $U_{\mathbb{S}}$ creates coequalizers of parallel pairs $f, g : (X, r) \to (Y, s)$ in $\mathbb{S}$-$\mathbf{Alg}$ for which $f, g : X \to Y$ has an absolute coequalizer, say $e : Y \twoheadrightarrow Z$, in $\mathscr{C}$. In this case then, Σe is a coequalizer of $\Sigma f, \Sigma g$ and Γe is a coequalizer of $\Gamma f, \Gamma g$, so that we have the following situation

$$\begin{tikzcd}
\Sigma X \arrow[r, shift left, "\Sigma f"] \arrow[r, shift right, "\Sigma g"'] \arrow[d, "r"'] & \Sigma Y \arrow[r, two heads, "\Sigma e"] \arrow[d, "s"'] & \Sigma Z \arrow[d, dashed, "\exists!\, t"] \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, two heads, "e", "\text{coeq}"'] & Z \\
\Gamma X \arrow[u, "L(X{,}r)^\diamond = R(X{,}r)^\diamond"] \arrow[r, shift left, "\Gamma f"] \arrow[r, shift right, "\Gamma g"'] & \Gamma Y \arrow[u, "L(Y{,}s)^\diamond = R(Y{,}s)^\diamond"] \arrow[r, two heads, "\Gamma e"] & \Gamma Z \arrow[u, dashed, shift left, "L(Z{,}t)^\diamond"] \arrow[u, dashed, shift right, "R(Z{,}t)^\diamond"']
\end{tikzcd}$$

where the map t is the unique mediating map from the coequalizer Σe to the map $e \circ s$. Since both maps $L(Z, t)^\diamond$ and $R(Z, t)^\diamond$ are factors of the map $e \circ L(Y, s)^\diamond = e \circ R(Y, s)^\diamond$

through the coequalizer Γe, we have that $L(Z,t)^\diamond = R(Z,t)^\diamond$, that is to say, (Z,t) is an $\mathbb{S}$-algebra.

From the universal properties of e and Σe, one can easily show that $e : (Y,s) \to (Z,t)$ is a coequalizer of $f, g : (X,r) \to (Y,s)$ in Σ-**Alg**, and hence also in $\mathbb{S}$-**Alg**. □

The monadicity and cocompleteness of categories of algebras for inductive equational systems follows from Proposition 3.3.4 and the following lemma.

Lemma 3.3.5. *Let Σ be an endofunctor on a category $\mathscr{C}$. If Σ-algebraic coequalizers exist, then the category Σ-**Alg** has coequalizers.*

Proof. For a pair of Σ-algebra homomorphisms $f, g : (X,s) \to (Y,t)$, a Σ-algebraic coequalizer of the maps $f, g : X \to Y$ into the Σ-algebra (Y,t) is, by definition, a coequalizer of the homomorphisms f, g in Σ-**Alg**. □

Theorem 3.3.6. *For an inductive equational system $\mathbb{S}$ on a category $\mathscr{C}$, the forgetful functor $U_\mathbb{S} : \mathbb{S}$-**Alg** $\to \mathscr{C}$ is monadic and the category $\mathbb{S}$-**Alg** is cocomplete.*

Proof. By Theorem 3.2.1, $U_\mathbb{S}$ has a left adjoint and thus, by Proposition 3.3.4, is monadic. Furthermore, $\mathbb{S}$-**Alg** has coequalizers since, by Theorem 3.2.8, it is a full reflective subcategory of Σ-**Alg** which, by Corollary 3.2.5 and Lemma 3.3.5, has coequalizers. Being monadic over a cocomplete category and having coequalizers, $\mathbb{S}$-**Alg** is cocomplete (see *e.g.* [Borceux 1994, Proposition 4.3.4]). □

3.3.2 Properties of representing monads

We show that representing monads of inductive equational systems are ω-cocontinuous and epicontinuous.

Cocontinuity. We show that in general the colimit-preservation properties of the functorial signature and functorial context of an equational system on a cocomplete category are inherited by its representing monad. It follows as a corollary that representing monads of inductive equational systems are ω-cocontinuous.

Recall that a diagram in a category $\mathscr{C}$ is a functor from a small category to $\mathscr{C}$. We say that a class $\mathcal{K}$ of diagrams in $\mathscr{C}$ is closed under an endofunctor F on $\mathscr{C}$ if the diagram $F\,I : \mathbb{I} \to \mathscr{C}$ is in $\mathcal{K}$ for every diagram $I : \mathbb{I} \to \mathscr{C}$ in $\mathcal{K}$.

Proposition 3.3.7. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system with representing monad (T, η, μ). For $\mathcal{K}$ a class of diagrams in $\mathscr{C}$ closed under T, if $\mathscr{C}$ has colimits of diagrams in $\mathcal{K}$ and the endofunctors Σ and Γ preserve them, then so does the endofunctor T.*

Proof. For a diagram $I : \mathbb{I} \to \mathscr{C}$ in $\mathcal{K}$, let $\{\lambda_i : Ii \to \operatorname{colim} I\}_{i\in\mathbb{I}}$ and $\{\delta_i : TIi \to \operatorname{colim} TI\}_{i\in\mathbb{I}}$ be colimiting cones. We show that the cones $\{\, T\lambda_i \,\}_{i\in\mathbb{I}}$ and $\{\, \delta_i \,\}_{i\in\mathbb{I}}$ are isomorphic. To this end, we construct an inverse $q : T(\operatorname{colim} I) \to \operatorname{colim} TI$ to the mediating map $p : \operatorname{colim} TI \to T(\operatorname{colim} I)$ from $\{\, \delta_i \,\}_{i\in\mathbb{I}}$ to $\{\, T\lambda_i \,\}_{i\in\mathbb{I}}$ as follows.

For every object X in $\mathscr{C}$, let $(TX, \tau_X : \Sigma TX \to TX)$ be the free $\mathbb{S}$-algebra on X induced by the left adjoint to $U_{\mathbb{S}}$. As the family $\tau = \{\, \tau_X : \Sigma TX \to TX \,\}_{X\in\mathscr{C}}$ is natural, the family $\{\, \delta_i \circ \tau_{Ii} : \Sigma TIi \to \operatorname{colim} TI \,\}_{i\in\mathbb{I}}$ is a cone and, as $\{\, \Sigma\delta_i \,\}_{i\in\mathbb{I}}$ is colimiting, we have a unique Σ-algebra structure map t on $\operatorname{colim} TI$ such that the diagram on the top below commutes for every $i \in \mathbb{I}$.

$$\begin{tikzcd}
\Sigma TIi \arrow[r, "\Sigma\delta_i"] \arrow[d, "\tau_{Ii}"'] & \Sigma(\operatorname{colim} TI) \arrow[d, dashed, "\exists!\, t"] \\
TIi \arrow[r, "\delta_i"] & \operatorname{colim} TI \\
\Gamma TIi \arrow[r, "\Gamma\delta_i"] \arrow[u, "L(TIi,\tau_{Ii})^\diamond = R(TIi,\tau_{Ii})^\diamond"] & \Gamma(\operatorname{colim} TI) \arrow[u, shift left, dashed, "R(\operatorname{colim} TI,t)^\diamond"'] \arrow[u, shift right, dashed, "L(\operatorname{colim} TI,t)^\diamond"]
\end{tikzcd}$$

Furthermore, the Σ-algebra $(\operatorname{colim} TI, t)$ is an $\mathbb{S}$-algebra; since $\{\, \Gamma\delta_i \,\}_{i\in\mathbb{I}}$ is colimiting and $L(\operatorname{colim} TI, t)^\diamond \circ \Gamma\delta_i = R(\operatorname{colim} TI, t)^\diamond \circ \Gamma\delta_i$ for all $i \in \mathbb{I}$.

By the universal property of free algebras, we define $q : T(\operatorname{colim} I) \to \operatorname{colim} TI$ as the unique map making the following diagram commutative:

$$\begin{tikzcd}
\Sigma T(\operatorname{colim} I) \arrow[r, dashed, "\Sigma q"] \arrow[d, "\tau_{\operatorname{colim} I}"'] & \Sigma(\operatorname{colim} TI) \arrow[d, "t"] \\
T(\operatorname{colim} I) \arrow[r, dashed, "\exists!\, q"] & \operatorname{colim} TI \\
\operatorname{colim} I \arrow[u, "\eta_{\operatorname{colim} I}"] \arrow[ur, "\operatorname{colim} \eta_I"']
\end{tikzcd}$$

This map is a morphism between the cones $\{\, T\lambda_i \,\}_{i\in\mathbb{I}}$ and $\{\, \delta_i \,\}_{i\in\mathbb{I}}$; as follows from the commutative diagrams below

$$\begin{tikzcd}
\Sigma TIi \arrow[r, "\Sigma T\lambda_i"] \arrow[d, "\tau_{Ii}"'] & \Sigma T(\operatorname{colim} I) \arrow[r, "\Sigma q"] \arrow[d, "\tau_{\operatorname{colim} I}"'] & \Sigma(\operatorname{colim} TI) \arrow[d, "t"] \\
TIi \arrow[r, "T\lambda_i"] & T(\operatorname{colim} I) \arrow[r, "q"] & \operatorname{colim} TI \\
Ii \arrow[u, "\eta_{Ii}"] \arrow[r, "\lambda_i"] & \operatorname{colim} I \arrow[u, "\eta_{\operatorname{colim} I}"'] \arrow[ur, "\operatorname{colim} \eta_I"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma TIi \arrow[rr, "\Sigma\delta_i"] \arrow[d, "\tau_{Ii}"'] & & \Sigma(\operatorname{colim} TI) \arrow[d, "t"] \\
TIi \arrow[rr, "\delta_i"] & & \operatorname{colim} TI \\
Ii \arrow[u, "\eta_{Ii}"] \arrow[r, "\lambda_i"] & \operatorname{colim} I \arrow[ur, "\operatorname{colim} \eta_I"'] &
\end{tikzcd}$$

by the universal property of free algebras.

The endomap $q \circ p$ on $\operatorname{colim} TI$ is the identity, as it is an endomap on a colimiting cone. That the endomap $p \circ q$ on $T(\operatorname{colim} I)$ is the identity follows from the commutativity of the diagram below

$$\begin{tikzcd}
\Sigma T(\operatorname{colim} I) \arrow[r, "\Sigma q"] \arrow[d, "\tau_{\operatorname{colim} I}"'] & \Sigma(\operatorname{colim} TI) \arrow[r, "\Sigma p"] \arrow[d, "t"] \arrow[dr, phantom, "(\mathsf{B})"] & \Sigma T(\operatorname{colim} I) \arrow[d, "\tau_{\operatorname{colim} I}"] \\
T(\operatorname{colim} I) \arrow[r, "q"] & \operatorname{colim} TI \arrow[r, "p"] & T(\operatorname{colim} I) \\
\operatorname{colim} I \arrow[u, "\eta_{\operatorname{colim} I}"] \arrow[urr, "\eta_{\operatorname{colim} I}"'] \arrow[ur, phantom, "(\mathsf{A})"] & &
\end{tikzcd}$$

by the universal property of free algebras. The commutativity of the diagram (A) above follows from the commutativity of the following diagram for each $i \in \mathbb{I}$

$$\begin{tikzcd}
Ii \arrow[r, "\lambda_i"] \arrow[dr, "\eta_{Ii}"'] \arrow[dd, "\lambda_i"'] & \operatorname{colim} I \arrow[r, "\eta_{\operatorname{colim} I}"] \arrow[dr, "\operatorname{colim} \eta_I"'] & T(\operatorname{colim} I) \arrow[d, "q"] \\
& TIi \arrow[r, "\delta_i"] \arrow[dr, "T\lambda_i"'] & \operatorname{colim} TI \arrow[d, "p"] \\
\operatorname{colim} I \arrow[rr, "\eta_{\operatorname{colim} I}"'] & & T(\operatorname{colim} I)
\end{tikzcd}$$

because $\{ \lambda_i \}_{i \in \mathbb{I}}$ is a colimiting cone. The commutativity of diagram (B) above follows from the commutativity of the following diagram for each $i \in \mathbb{I}$

$$\begin{tikzcd}
\Sigma TIi \arrow[rr, "\Sigma \delta_i"] \arrow[drr, "\Sigma T\lambda_i"] \arrow[ddr, "\tau_{Ii}"] \arrow[ddd, "\Sigma \delta_i"'] & & \Sigma(\operatorname{colim} TI) \arrow[d, "\Sigma p"] \\
& & \Sigma T(\operatorname{colim} I) \arrow[dd, "\tau_{\operatorname{colim} I}"] \\
& TIi \arrow[d, "\delta_i"'] \arrow[dr, "T\lambda_i"] & \\
\Sigma(\operatorname{colim} TI) \arrow[r, "t"'] & \operatorname{colim} TI \arrow[r, "p"'] & T(\operatorname{colim} I)
\end{tikzcd}$$

because $\{ \Sigma\delta_i \}_{i \in \mathbb{I}}$ is a colimiting cone. □

When considered for the class of all ω-chains, the above theorem yields the following corollary.

Corollary 3.3.8. *Representing monads of inductive equational systems are ω-cocontinuous.*

Epicontinuity. For an equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ with representing monad (T, η, μ), it follows from Proposition 3.3.7 that if Σ and Γ preserve cokernel pairs (*viz.*, pushouts of spans with identical legs) then so does T; so that, in particular, it also preserves epimorphisms. However, under the free constructions of Section 3.2, one can directly obtain the epicontinuity of representing monads of inductive equational systems.

Theorem 3.3.9. *Representing monads of inductive equational systems are epicontinuous.*

Proof. For an inductive equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$, we recall the free constructions (3.3) and (3.4), which jointly yield left adjoints to the forgetful functors $U_\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ and $U_\mathbb{S} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$. We write (T, η, μ) for the monad induced by the former adjunction and $(\widetilde{T}, \widetilde{\eta}, \widetilde{\mu})$ for the representing monad of $\mathbb{S}$ induced by the latter adjunction. We show that the endofunctor $\widetilde{T}$ is epicontinuous.

We consider, for each epimorphism $f : X \twoheadrightarrow Y$ in $\mathscr{C}$, the constructions of the map $Tf : TX \to TY$ induced from (3.3), and of the map $\widetilde{T}f : \widetilde{T}X \to \widetilde{T}Y$ induced from (3.4):

$$
\begin{tikzcd}
0 \arrow[r, "!"] \arrow[d, two heads, "0"] & X + \Sigma 0 \arrow[r, "X+\Sigma!"] \arrow[d, two heads, "f+\Sigma 0"] & X + \Sigma(X + \Sigma 0) \arrow[r] \arrow[d, two heads, "f+\Sigma(f+\Sigma 0)"] & \cdots\cdots & TX \arrow[r, two heads, "q_X"] \arrow[d, dashed, two heads, "\exists!\, Tf"] & \widetilde{T}X \arrow[d, dashed, "\exists!\, \widetilde{T}f"] \\
0 \arrow[r, "!"] & Y + \Sigma 0 \arrow[r, "Y+\Sigma!"] & Y + \Sigma(Y + \Sigma 0) \arrow[r] & \cdots\cdots & \underset{\mathsf{colim}}{TY} \arrow[r, two heads, "q_Y"] & \widetilde{T}Y
\end{tikzcd}
$$

Since Σ is epicontinuous, the vertical maps 0, $f + \Sigma 0$, $\ldots$ are inductively shown to be epimorphic, and thus so is the map Tf. As the quotient map q_Y is epimorphic, then so is the composite $q_Y \circ Tf = \widetilde{T}f \circ q_X$, and thus the map $\widetilde{T}f$ is epimorphic. $\square$

3.3.3 Properties of categories of inductive equational systems

The cocompleteness of categories of inductive equational systems follows as a corollary of the following more general result.

Proposition 3.3.10. *For a cocomplete category $\mathscr{C}$, the category $\mathbf{ES}(\mathscr{C})$ of equational systems on $\mathscr{C}$ has*

(i) small coproducts; and

(ii) coequalizers of parallel pairs $\mathbb{S}_1 \rightrightarrows \mathbb{S}_2$ for which $\mathbb{S}_2$ has a representing monad.

Proof. Recall from Section 2.2.3 that $\mathbf{ES}(\mathscr{C})$ is a full subcategory of the cocomplete category $(\mathbf{CAT}/\mathscr{C})^{\mathrm{op}}$ via the embedding sending an equational system $\mathbb{S}$ to the pair $(\mathbb{S}\text{-}\mathbf{Alg}, U_\mathbb{S} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C})$.

For (i), given a family of equational systems

$$\{\, \mathbb{S}_i = (\mathscr{C} : \Sigma_i \rhd \Gamma_i \vdash L_i \equiv R_i) \,\}_{i \in I} \ ,$$

define the equational system

$$\textstyle\coprod_{i \in I} \mathbb{S}_i = \big(\mathscr{C} : \coprod_{i \in I} \Sigma_i \rhd \coprod_{i \in I} \Gamma_i \vdash [L_i\, U_i]_{i \in I} \equiv [R_i\, U_i]_{i \in I}\big) \tag{3.5}$$

where U_i denotes the forgetful functor $(\coprod_{i \in I} \Sigma_i)\text{-}\mathbf{Alg} \to \Sigma_i\text{-}\mathbf{Alg}$. It follows that the system $\coprod_{i \in I} \mathbb{S}_i$ is a coproduct of the family $\{\, \mathbb{S}_i \,\}_{i \in I}$ in $\mathbf{ES}(\mathscr{C})$, as $(\coprod_{i \in I} \mathbb{S}_i)\text{-}\mathbf{Alg}$ is a product of $\{\, \mathbb{S}_i\text{-}\mathbf{Alg} \,\}_{i \in I}$ in the category $\mathbf{CAT}/\mathscr{C}$.

For (ii), let $\mathbb{S}_1 = (\mathscr{C} : \Sigma_1 \rhd \Gamma_1 \vdash L_1 \equiv R_1)$, $\mathbb{S}_2 = (\mathscr{C} : \Sigma_2 \rhd \Gamma_2 \vdash L_2 \equiv R_2)$ be equational systems such that $\mathbb{S}_2$ has a representing monad $\mathbf{T}_2 = (T_2, \eta_2, \mu_2)$, and let

$F, G : \mathbb{S}_1 \to \mathbb{S}_2$ be a pair of morphisms between them. By definition, the morphisms F, G are functors $F, G : \mathbb{S}_2\text{-}\mathbf{Alg} \to \mathbb{S}_1\text{-}\mathbf{Alg}$ over $\mathscr{C}$ and we have the following situation

$$\mathscr{C}^{\mathbf{T}_2} \xrightarrow[\cong]{I_2} \mathbb{S}_2\text{-}\mathbf{Alg} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{S}_1\text{-}\mathbf{Alg} \overset{J_1}{\hookrightarrow} \Sigma_1\text{-}\mathbf{Alg}$$

where $\mathscr{C}^{\mathbf{T}_2}$ is the category of Eilenberg-Moore algebras for $\mathbf{T}_2$ and I_2 is the inverse of the comparison isomorphism given by the monadicity of $\mathbb{S}_2\text{-}\mathbf{Alg}$ over $\mathscr{C}$. Recalling the concept of monadic equational system from Section 2.3, we consider the monadic system

$$\mathbb{S}_3 = (\mathscr{C} : \mathbf{T}_2 \rhd \Sigma_1 \vdash J_1\, F\, I_2 \equiv J_1\, G\, I_2).$$

By definition, the embedding $J_3 : \mathbb{S}_3\text{-}\mathbf{Alg} \hookrightarrow \mathscr{C}^{\mathbf{T}_2}$ is an equalizer of $J_1\, F\, I_2$ and $J_1\, G\, I_2$ in $\mathbf{CAT}/\mathscr{C}$. As I_2 is an isomorphism and J_1 is a monomorphism in $\mathbf{CAT}/\mathscr{C}$, the functor $I_2\, J_3 : \mathbb{S}_3\text{-}\mathbf{Alg} \to \mathbb{S}_2\text{-}\mathbf{Alg}$ is an equalizer of F and G in $\mathbf{CAT}/\mathscr{C}$. As we have seen in Section 2.3, the monadic system $\mathbb{S}_3$ induces the equivalent equational system $\overline{\mathbb{S}_3}$ in the sense that $\mathbb{S}_3\text{-}\mathbf{Alg} = \overline{\mathbb{S}_3}\text{-}\mathbf{Alg}$. It thus follows that $\overline{\mathbb{S}_3}$ is a coequalizer of F, G in $\mathbf{ES}(\mathscr{C})$. Furthermore, from the definition of $\overline{\mathbb{S}_3}$, we note that

- the functorial signature of $\overline{\mathbb{S}_3}$ is the endofunctor T_2; and
- the functorial context of $\overline{\mathbb{S}_3}$ is the endofunctor $\mathrm{Id} + T_2\, T_2 + \Sigma_1$. (3.6)

□

Corollary 3.3.11. *For a cocomplete category $\mathscr{C}$, the full subcategory* $\mathbf{IndES}(\mathscr{C})$ *of* $\mathbf{ES}(\mathscr{C})$ *consisting of inductive equational systems is cocomplete.*

Proof. As every inductive equational system has a representing monad, it follows from Proposition 3.3.10 that small coproducts and coequalizers of inductive equational systems exist in $\mathbf{ES}(\mathscr{C})$. Moreover, that those colimits in $\mathbf{ES}(\mathscr{C})$ are also inductive follows from (3.5) and (3.6), and from the fact that representing monads of inductive equational systems are epicontinuous and ω-cocontinuous (Theorem 3.3.9 and Corollary 3.3.8). Thus, the category $\mathbf{IndES}(\mathscr{C})$ has small colimits. □

3.4 Examples

We revisit the examples of equational systems given in Section 2.5 in the light of the results of this chapter.

1. For the equational system $\mathbb{S}_\mathbb{T} = (\mathbf{Set} : \Sigma_\mathbb{T} \rhd \Gamma_\mathbb{T} \vdash L_\mathbb{T} \equiv R_\mathbb{T})$ representing an algebraic theory $\mathbb{T}$, the system $\mathbb{S}_\mathbb{T}$ is inductive, and one can apply Theorem 3.3.1 as follows: the category $\mathbb{S}_\mathbb{T}\text{-}\mathbf{Alg}$ is monadic over $\mathbf{Set}$ and cocomplete; and the free-algebra monad (*i.e.*, the representing monad of $\mathbb{S}_\mathbb{T}$) is epicontinuous and ω-cocontinuous. As the endofunctors $\Sigma_\mathbb{T}$ and $\Gamma_\mathbb{T}$ are finitary (*i.e.*, they preserve filtered colimits), by Proposition 3.3.7, the free-algebra monad is finitary.

2. For the equational system $\mathbb{S}_{\mathbb{T}} = (\mathscr{C}_0 : \Sigma_{\mathbb{T}} \rhd \Gamma_{\mathbb{T}} \vdash L_{\mathbb{T}} \equiv R_{\mathbb{T}})$ representing an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, \Sigma, E)$, the functors $\Sigma_{\mathbb{T}}$ and $\Gamma_{\mathbb{T}}$ are ω-cocontinuous but need not be epicontinuous (see Remark 3.4.1 below for a counter example). Thus one cannot apply the theory of this chapter. However, in Section 4.3, we will show the cocompleteness and monadicity of $\mathbb{S}_{\mathbb{T}}$-**Alg** using a more general theory developed in Chapter 4.

   Remark 3.4.1. The category **Nom** of nominal sets (see *e.g.* [Gabbay and Pitts 1999, 2001] and Section 8.2.1) is a suitable **Set**-enriched category for defining enriched algebraic theories, as it is locally finitely presentable. Consider the enriched algebraic theory $\mathbb{T} = (\mathbf{Nom}, \Sigma, \emptyset)$ with Σ the signature consisting of only an operator of arity $\mathbb{A}$ and coarity 1, for $\mathbb{A}$ the nominal set of atoms (see Section 8.2.1). The functorial signature $\Sigma_{\mathbb{T}}$ is then given by

$$\Sigma_{\mathbb{T}}(X) \;=\; \mathbf{Nom}(\mathbb{A}, X) \otimes 1 \;=\; \coprod_{f \in \mathbf{Nom}(\mathbb{A}, X)} 1 \; .$$

   However, the endofunctor $\Sigma_{\mathbb{T}}$ is not epicontinuous: for the epimorphism $! : \mathbb{A} \# \mathbb{A} \to 1$, where $\mathbb{A} \# \mathbb{A}$ is the nominal subset of $\mathbb{A} \times \mathbb{A}$ with underlying set given by $\{ (a, b) \in \mathbb{A} \times \mathbb{A} \mid a \neq b \}$, the morphism $\Sigma_{\mathbb{T}}(!)$ is not epimorphic since $\mathbf{Nom}(\mathbb{A}, \mathbb{A} \# \mathbb{A})$ is the empty set and $\mathbf{Nom}(\mathbb{A}, 1)$ is a singleton set.

3. We may apply Theorem 3.3.1 to the equational system $\mathbb{S}_{\mathbf{T}}$ representing a monad $\mathbf{T} = (T, \eta, \mu)$ on a cocomplete category $\mathscr{C}$ as follows. If T is epicontinuous and ω-cocontinuous, then $\mathbb{S}_{\mathbf{T}}$-**Alg** $\cong \mathscr{C}^{\mathbf{T}}$ is cocomplete.

   Moreover, for a cocomplete category $\mathscr{C}$, the category of monads on $\mathscr{C}$ preserving both epimorphisms and colimits of ω-chains is equivalent to the category $\mathbf{IndES}(\mathscr{C})$ through

   a) the embedding sending a monad $\mathbf{T}$ to the equational system $\mathbb{S}_{\mathbf{T}}$ (see item 3 of Section 2.5), and

   b) the embedding sending an inductive equational system to its representing monad.

   Thus, by Corollary 3.3.11, we can conclude that the category of monads on a cocomplete category that preserve both epimorphisms and colimits of ω-chains is cocomplete.

4. To the equational system $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$ of monoids in a cocomplete monoidal category $\mathscr{C}$, we can apply Theorem 3.3.1 as follows. If the tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is epicontinuous and ω-cocontinuous (as it happens, for instance, when it is biclosed), then $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$-**Alg** (*i.e.*, the category of monoids in $\mathscr{C}$) is cocomplete and monadic over $\mathscr{C}$, and the free-monoid monad (*i.e.*, the representing monad of $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$) is epicontinuous and ω-cocontinuous.

5. To the equational cosystem $\mathbb{S}_{\mathrm{CoMon}(\mathscr{C})}$ of comonoids in a complete monoidal category $\mathscr{C}$, we apply the dual version of Theorem 3.3.1 as follows. If the tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ preserves monomorphisms and limits of ω-cochains, then $\mathbb{S}_{\mathrm{CoMon}(\mathscr{C})}$-**CoAlg** (*i.e.*, the category of comonoids in $\mathscr{C}$) is complete and comonadic over $\mathscr{C}$, and the cofree-comonoid comonad (*i.e.*, the representing comonad of $\mathbb{S}_{\mathrm{CoMon}(\mathscr{C})}$) preserves monomorphisms and limits of ω-cochains.

Chapter 4

General theory of equational systems

We seek more general conditions on equational systems for admitting the properties that we have discussed in Chapter 3. In this respect, we generalize the inductiveness condition as follows. Note that the notion of inductiveness given in Chapter 3 amounts to that of ω-inductiveness given below.

Definition 4.0.2. An equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ is said to be *κ-finitary*, for κ an infinite limit ordinal, if the category $\mathscr{C}$ is cocomplete and both functors Σ and Γ are κ-cocontinuous. Such an equational system is said to be *κ-inductive* if furthermore both functors Σ and Γ are epicontinuous.

Main results of this chapter are summarised as follows. For every κ-finitary equational system $\mathbb{S}$, the following hold.

- The category $\mathbb{S}$-**Alg** of $\mathbb{S}$-algebras is cocomplete and monadic over the base category.
- The representing monad of $\mathbb{S}$ is κ-cocontinuous.

Furthermore, when $\mathbb{S}$ is κ-inductive, we additionally have the following.

- Free $\mathbb{S}$-algebras are constructed in $\kappa + \kappa$ steps.
- The representing monad of $\mathbb{S}$ is epicontinuous.

We also have the following result.

- The categories κ-**FinES**$(\mathscr{C})$ and κ-**IndES**$(\mathscr{C})$ of κ-finitary and κ-inductive equational systems on a cocomplete category $\mathscr{C}$ are cocomplete.

Note that throughout the chapter we develop more technical and general conditions than the finitariness and inductiveness conditions above.

We follow the development of Chapter 3, extending the inductive constructions to the transfinite case (see Section 4.1 and Section 4.2). We conclude the chapter by revisiting the example equational systems given in Section 2.5 in the light of these results (see Section 4.3).

4.1 Transfinite free constructions for equational systems

This technical section extends the inductive constructions and results of Section 3.2 to the transfinite case. Overall, by means of Corollaries 4.1.12 and 4.1.13, the following theorems are established.

Theorem 4.1.1. *For a κ-finitary equational system $\mathbb{S}$ on a category $\mathscr{C}$, the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint. Furthermore, if the system $\mathbb{S}$ is κ-inductive, free $\mathbb{S}$-algebras on objects in $\mathscr{C}$ can be constructed in $\kappa + \kappa$ steps, as in the diagram (4.1) followed by (4.2).*

Theorem 4.1.2. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system with $\mathscr{C}$ cocomplete. If $\mathscr{C}$ has no transfinite chain of proper epimorphisms, and Σ is epicontinuous and κ-cocontinuous for some infinite limit ordinal κ, then the forgetful functor $\mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint.*

Remark 4.1.3. In Theorem 4.1.2, we take a transfinite chain in a category $\mathscr{C}$ to be an $\mathbf{Ord}$-indexed diagram (*i.e.*, a functor from $\mathbf{Ord}$ to $\mathscr{C}$) for $\mathbf{Ord}$ the large linear order of ordinals. Main examples of categories with no transfinite chain of proper epimorphisms are those that are well-copowered. Recall that a category is called *well-copowered* if, for each object, the collection of its quotient objects is a set.

Analogously to the development in Section 3.2, we consider the construction of algebraic coequalizers, free Σ-algebras, and free $\mathbb{S}$-algebras in turn.

4.1.1 Algebra cospans and algebraic coequalizers

We generalize the construction (3.1) of Section 3.2.1 for reflecting algebra cospans into algebras.

Let Σ be an endofunctor on a category $\mathscr{C}$ and $(c : Z \to Z' \leftarrow \Sigma Z : t)$ a Σ-algebra cospan. For κ an ordinal, we proceed to consider a (possibly transfinite) construction as depicted below

$$\begin{tikzcd}
\Sigma Z \arrow[r, "\Sigma c"] \arrow[ddr, "t"] & \Sigma Z' \arrow[r, "\Sigma c_{1,2}"] \arrow[ddr, "t_1"] \arrow[d, phantom, "\text{po}"] & \Sigma Z_2 \arrow[rr, bend left, "\Sigma c_{2,\omega}"] \arrow[rr, phantom, "\cdots\cdots"] \arrow[dr, bend right, "c^*_{2,\omega}"'] & & \Sigma Z_\omega \arrow[r, "\Sigma c_{\omega,\omega+1}"] \arrow[ddr, "t_\omega"] & \Sigma Z_{\omega+1} & \cdots & \Sigma Z_\kappa \arrow[ddr, "t_\kappa"] & \\
& & & \underset{\text{colim}}{Z^*_\omega} \arrow[ur, "c^*_\omega"] \arrow[dr, "t^*_\omega"] \arrow[r, phantom, "\text{po}"] & \; & & & & \\
Z \arrow[r, "c"'] & Z' \arrow[r, "c_{1,2}"'] & Z_2 \arrow[rr, bend right, "c_{2,\omega}"'] \arrow[rr, phantom, "\cdots\cdots"] & & \underset{\text{colim}}{Z_\omega} \arrow[r, "c_{\omega,\omega+1}"'] & Z_{\omega+1} & \cdots & Z_\kappa \arrow[r, "c_{\kappa,\kappa+1}"'] & Z_{\kappa+1}
\end{tikzcd} \qquad (*)$$

yielding

- a chain $\{ c_{\alpha,\beta} : Z_\alpha \to Z_\beta \}_{\alpha \leq \beta \leq \kappa+1}$ (with $c_{0,1} = c$), and

- morphisms $\{ t_\alpha : \Sigma Z_\alpha \to Z_{\alpha+1} \}_{\alpha \leq \kappa}$ (with $t_0 = t$)

such that

$$\begin{tikzcd} \Sigma Z_\alpha \arrow[rr, "\Sigma c_{\alpha,\beta}"] \arrow[dr, "t_\alpha"] & & \Sigma Z_\beta \arrow[dr, "t_\beta"] & \\ & Z_{\alpha+1} \arrow[rr, "c_{\alpha+1,\beta+1}"'] & & Z_{\beta+1} \end{tikzcd}$$

commutes.

Precisely, the definitions are as follows: for $\lambda \leq \kappa$,

- when $\lambda = 0$,
  $Z_\lambda \overset{c_{\lambda,\lambda+1}}{\longrightarrow} Z_{\lambda+1} \overset{t_\lambda}{\longleftarrow} \Sigma Z_\lambda$ is $Z \overset{c}{\longrightarrow} Z' \overset{t}{\longleftarrow} \Sigma Z$;

- when λ is a successor ordinal $\alpha + 1$,
  $Z_\lambda \overset{c_{\lambda,\lambda+1}}{\longrightarrow} Z_{\lambda+1} \overset{t_\lambda}{\longleftarrow} \Sigma Z_\lambda$ is a pushout of $Z_{\alpha+1} \overset{t_\alpha}{\longleftarrow} \Sigma Z_\alpha \overset{\Sigma c_{\alpha,\alpha+1}}{\longrightarrow} \Sigma Z_{\alpha+1}$; and

- when λ is a limit ordinal,
  $Z_\lambda \overset{c_{\lambda,\lambda+1}}{\longrightarrow} Z_{\lambda+1} \overset{t_\lambda}{\longleftarrow} \Sigma Z_\lambda$ is a pushout of $Z_\lambda \overset{t^*_\lambda}{\longleftarrow} Z^*_\lambda \overset{c^*_\lambda}{\longrightarrow} \Sigma Z_\lambda$, where
  - $\{ c_{\alpha,\lambda} : Z_\alpha \to Z_\lambda \}_{\alpha<\lambda}$ and $\{ c^*_{\alpha,\lambda} : \Sigma Z_\alpha \to Z^*_\lambda \}_{\alpha<\lambda}$ are respectively colimits of the λ-chains $\{ c_{\alpha,\beta} \}_{\alpha\leq\beta<\lambda}$ and $\{ \Sigma c_{\alpha,\beta} \}_{\alpha\leq\beta<\lambda}$; and
  - c^*_λ and t^*_λ are respectively the mediating maps from the colimiting cone $\{ c^*_{\alpha,\lambda} \}_{\alpha<\lambda}$ to the cones $\{ \Sigma c_{\alpha,\lambda} \}_{\alpha<\lambda}$ and $\{ c_{\alpha+1,\lambda} \circ t_\alpha \}_{\alpha<\lambda}$ of the λ-chain $\{ \Sigma c_{\alpha,\beta} \}_{\alpha\leq\beta<\lambda}$.

Definition 4.1.4. Whenever the construction $(*)$ above can be performed for the ordinal κ, we say that it *reaches* κ. Furthermore, we say that the construction $(*)$ *stops* if it does so at some ordinal κ in the sense that it reaches κ and the map $c_{\kappa,\kappa+1} : Z_\kappa \to Z_{\kappa+1}$ is an isomorphism.

We now show that if the construction $(*)$ stops, then it reflects the Σ-algebra cospan $(c : Z \to Z' \leftarrow \Sigma Z : t)$ into a Σ-algebra.

Theorem 4.1.5. *Let Σ be an endofunctor on a category $\mathscr{C}$. For a Σ-algebra cospan $(c : Z \to Z' \leftarrow \Sigma Z : t)$, if the construction $(*)$ for it stops, then a free Σ-algebra on it exists. If, in addition, the endofunctor Σ is epicontinuous and the map c is epimorphic in $\mathscr{C}$, then the two components of the universal map from the Σ-algebra cospan to the free Σ-algebra are epimorphic in $\mathscr{C}$.*

Proof. Let $(Z \overset{c}{\to} Z' \overset{t}{\leftarrow} \Sigma Z)$ be a Σ-algebra cospan and assume that the construction $(*)$ for it stops at an ordinal κ. We claim that the Σ-algebra $(Z_\kappa, (c_{\kappa,\kappa+1})^{-1} \circ t_\kappa : \Sigma Z_\kappa \to Z_\kappa)$ is free over $(c : Z \to Z' \leftarrow \Sigma Z : t)$. Indeed, we show that

$$(c_{0,\kappa}, c_{1,\kappa}) : (Z_0 \to Z_1 \leftarrow \Sigma Z_0) \longrightarrow (Z_\kappa \overset{\mathrm{id}}{\to} Z_\kappa \leftarrow \Sigma Z_\kappa)$$

is a universal map in Σ-**AlgCoSp**.

First, note that $(c_{0,\kappa}, c_{1,\kappa})$ is indeed a map in Σ-**AlgCoSp**; as we have that

$$(c_{\kappa,\kappa+1})^{-1} \circ t_\kappa \circ \Sigma c_{0,\kappa} = (c_{\kappa,\kappa+1})^{-1} \circ c_{1,\kappa+1} \circ t_0 = c_{1,\kappa} \circ t_0 \,.$$

Second, consider a map $(h, h') : (Z \xrightarrow{c} Z' \xleftarrow{t} \Sigma Z) \longrightarrow (W \xrightarrow{\mathrm{id}} W \xleftarrow{u} \Sigma W)$ in Σ-**AlgCoSp** and perform the following construction:

$$\begin{tikzcd}
\Sigma Z \arrow[r, "\Sigma c"] \arrow[dddr, "t"] \arrow[drrrrrrrr, "\Sigma h"] & \Sigma Z' \arrow[r, "\Sigma c_{1,2}"] \arrow[dddr, "t_1"] \arrow[drrrrrrr, "\Sigma h'"] & \Sigma Z_2 \arrow[rr, bend left, "\Sigma c_{2,\omega}"] \arrow[dr, "c^*_{2,\omega}"'] \arrow[drrrrrr, "\Sigma h_2"] & \cdots\cdots & \Sigma Z_\omega \arrow[r, "\Sigma c_{\omega,\omega+1}"] \arrow[dddr, "t_\omega"] \arrow[drrrr, "\Sigma h_\omega"] & \Sigma Z_{\omega+1} \arrow[drrr, "\Sigma h_{\omega+1}"] & \cdots & \Sigma Z_\kappa \arrow[dd, "t_\kappa"] \arrow[dr, "\Sigma h_\kappa"] & \\
& & & Z^*_\omega \arrow[ur, "c^*_\omega"] \arrow[ddr, "t^*_\omega"] & & & & & \Sigma W \arrow[ddd, "u"] \\
& & & & & & & Z_{\kappa+1} \arrow[ddr, "h_{\kappa+1}"] & \\
Z \arrow[r, "c"] \arrow[drrrrrrrr, "h"] & Z' \arrow[r, "c_{1,2}"] \arrow[drrrrrrr, "h'"] & Z_2 \arrow[rr, bend left, "c_{2,\omega}"] \arrow[drrrrrr, "h_2"] & \cdots\cdots & Z_\omega \arrow[r, "c_{\omega,\omega+1}"] \arrow[drrrr, "h_\omega"] & Z_{\omega+1} \arrow[drrr, "h_{\omega+1}"] & \cdots & Z_\kappa \arrow[u, "c_{\kappa,\kappa+1}", "\cong"'] \arrow[dr, "h_\kappa"] & \\
& & & & & & & & W
\end{tikzcd}$$

where

- for $\lambda = 0$,
  h_λ is h and $h_{\lambda+1}$ is h';

- for a successor ordinal $\lambda = \alpha + 1$,
  h_λ is $h_{\alpha+1}$, and $h_{\lambda+1}$ is the mediating map from the pushout $Z_{\lambda+1}$ to W with respect to the cone $(Z_\lambda \xrightarrow{h_\lambda} W \xleftarrow{u \circ \Sigma h_\lambda} \Sigma Z_\lambda)$ of the span $(Z_{\alpha+1} \xleftarrow{t_\alpha} \Sigma Z_\alpha \xrightarrow{\Sigma c_{\alpha,\alpha+1}} \Sigma Z_{\alpha+1})$; and

- for a limit ordinal λ,
  h_λ is the mediating map from the colimit Z_λ to W with respect to the cone $\{\, h_\alpha \,\}_{\alpha<\lambda}$ of the λ-chain $\{\, c_{\alpha,\beta} \,\}_{\alpha\leq\beta<\lambda}$, and $h_{\lambda+1}$ is the mediating map from the pushout $Z_{\lambda+1}$ to W with respect to the cone $(h_\lambda : Z_\lambda \to W \leftarrow \Sigma Z_\lambda : u \circ \Sigma h_\lambda)$ of the span $(t^*_\lambda : Z_\lambda \leftarrow Z^*_\lambda \to \Sigma Z_\lambda : c^*_\lambda)$.

As $h_\kappa \circ (c_{\kappa,\kappa+1})^{-1} \circ t_\kappa = h_{\kappa+1} \circ t_\kappa = u \circ \Sigma h_\kappa$, it follows that h_κ is a Σ-algebra homomorphism $(Z_\kappa, (c_{\kappa,\kappa+1})^{-1} \circ t_\kappa) \to (W, u)$. Hence, (h, h') factors as the composite $(h_\kappa, h_\kappa) \circ (c_{0,\kappa}, c_{1,\kappa})$ in Σ-**AlgCoSp**.

We finally establish the uniqueness of such factorizations. For any homomorphism $g : (Z_\kappa, (c_{\kappa,\kappa+1})^{-1} \circ t_\kappa) \to (W, u)$ such that $g \circ c_{1,\kappa} = h'$ it follows by transfinite induction that $g \circ c_{\alpha,\kappa} = h_\alpha$ for all $\alpha \leq \kappa$, and hence that $g = h_\kappa$.

If Σ is epicontinuous and c is an epimorphism in $\mathscr{C}$, then, by transfinite induction, the morphisms $c_{\alpha,\beta}$ and $\Sigma c_{\alpha,\beta}$ are shown to be epimorphic in $\mathscr{C}$ for all ordinals $\alpha \leq \beta \leq \kappa$. Hence this is the case for $c_{0,\kappa}$ and $c_{1,\kappa}$. □

A construction of algebraic coequalizers follows as a corollary.

Corollary 4.1.6. *Let Σ be an endofunctor on a category $\mathscr{C}$ with coequalizers. If the construction $(*)$ stops for every Σ-algebra cospan $(c : Z \to Z_1 \leftarrow \Sigma Z : t)$ with c epimorphic in $\mathscr{C}$, then Σ-algebraic coequalizers exist. If, in addition, Σ is epicontinuous then Σ-algebraic coequalizers are epimorphic in $\mathscr{C}$.*

Proof. Let $(Z, t : \Sigma Z \to Z)$ be a Σ-algebra and let l, r be a parallel pair into Z in $\mathscr{C}$. Consider a coequalizer $c : Z \twoheadrightarrow Z_1$ of l, r in $\mathscr{C}$ and the Σ-algebra cospan $(Z \xrightarrow{c} Z_1 \xleftarrow{c\circ t} \Sigma Z)$ as in the proof of Corollary 3.2.5:

$$\begin{tikzcd}
 & \Sigma Z \arrow[rr, "\Sigma z"] \arrow[d, "t"'] \arrow[dr, "c\circ t"] & & \Sigma Z' \arrow[d, "t'"] \\
Y \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & Z \arrow[r, two heads, "c", "\mathsf{coeq}"'] & Z_1 \arrow[r, "z_1"] \arrow[ur, phantom, "\mathsf{reflect}"] & Z'
\end{tikzcd}$$

As c is epimorphic, the construction $(*)$ stops for the Σ-algebra cospan $(Z \xrightarrow{c} Z_1 \xleftarrow{c\circ t} \Sigma Z)$ and thus, by Theorem 4.1.5, there exists a free Σ-algebra (Z', t') on the Σ-algebra cospan. Let $(z, z_1) : (Z \xrightarrow{c} Z_1 \xleftarrow{c\circ t} \Sigma Z) \longrightarrow (Z' \xrightarrow{\mathrm{id}} Z' \xleftarrow{t'} \Sigma Z')$ be the universal map. Then, the homomorphism $z = z_1 \circ c : (Z, t) \to (Z', t')$ is a Σ-algebraic coequalizer of l, r.

If Σ is epicontinuous, then by Theorem 4.1.5 the algebraic coequalizer z is epimorphic in $\mathscr{C}$ as so is c. $\square$

4.1.2 Construction of free algebras for endofunctors

The well-known transfinite construction of free algebras for endofunctors (see *e.g.* [Adámek 1974]) follows from Theorem 4.1.5 and the fact that the construction $(*)$ stops when $\mathscr{C}$ cocomplete and Σ κ-cocontinuous for an infinite limit ordinal κ (see Theorem 4.1.10).

Corollary 4.1.7. *For an endofunctor Σ on a category $\mathscr{C}$ with finite coproducts, let Σ_X, for $X \in \mathscr{C}$, be the endofunctor $X + \Sigma(-)$ on $\mathscr{C}$. For an object $X \in \mathscr{C}$, if the construction $(*)$ with respect to the endofunctor Σ_X for the initial Σ_X-algebra cospan $(0 \xrightarrow{!} \Sigma_X 0 \xleftarrow{\mathrm{id}} \Sigma_X 0)$ stops, then it yields an initial Σ_X-algebra whose Σ-algebra component is a free Σ-algebra on X.*

Note that in the above particular case of the construction $(*)$ depicted below, we have that $X_0 = 0$; that $X_{\alpha+1} = X + \Sigma X_\alpha$ for all successor ordinals $\alpha + 1$; and that X_λ is a colimit of the λ-chain $\{ c_{\alpha,\beta} \}_{\alpha\leq\beta<\lambda}$ for all limit ordinals λ.

$$\begin{tikzcd}
X+\Sigma 0 \arrow[r, "X+\Sigma(!)"] \arrow[ddr, "\mathrm{id}"] & X+\Sigma X_1 \arrow[r, "X+\Sigma c_{1,2}"] \arrow[ddr, "\mathrm{id}"] & X+\Sigma X_2 \arrow[rr, bend left, "X+\Sigma c_{2,\omega}"] \arrow[dr, "c_{3,\omega}"'] & \cdots & X+\Sigma X_\omega \arrow[r, "\Sigma c_{\omega,\omega+1}"] \arrow[ddr, "\mathrm{id}"] & X+\Sigma X_{\omega+1} & \cdots & X+\Sigma X_\kappa \arrow[d, "\mathrm{id}"'] \\
 & \mathsf{po} & & X_\omega\ \mathsf{colim} \arrow[ur, "c^*_\omega"'] \arrow[dr, "\mathrm{id}"] & \mathsf{po} & & & X_{\kappa+1} \arrow[d, "c^{-1}_{\kappa,\kappa+1}"', "\cong"] \\
0 \arrow[r, "!"'] & X_1 \arrow[r, "c_{1,2}"'] & X_2 \arrow[rr, bend right, "c_{2,\omega}"'] & \cdots & X_\omega\ \mathsf{colim} \arrow[r, "c_{\omega,\omega+1}"'] & X_{\omega+1} & \cdots & X_\kappa
\end{tikzcd} \tag{4.1}$$

4.1.3 Construction of free algebras for equational systems

We present a construction of free $\mathbb{S}$-algebras over Σ-algebras for $\mathbb{S}$ an equational system with functorial signature Σ, generalizing the construction (3.4) of Section 3.2.3.

Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system and (X, s) a Σ-algebra. For κ an ordinal, we proceed to consider a (possibly transfinite) construction as depicted below

$$
\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma e_{0,1}"] \arrow[dd, "s"] & \Sigma X_1 \arrow[rrr, "\Sigma e^\circ_{1,\omega}"] \arrow[rrrrr, bend left, "\Sigma e_{1,\omega}"] \arrow[drr, "e^*_{1,\omega}"'] \arrow[dd, "s_1"] & \cdots\cdots & & \Sigma X^\circ_\omega \arrow[rr, "\Sigma(e^\bullet_\omega \circ e^\circ_\omega)"] \arrow[ddr, "s^\circ_\omega"] & \arrow[dd, phantom, "\mathsf{reflect}"] & \Sigma X_\omega \arrow[dd, "s_\omega"] & \cdots & \Sigma X_\kappa \arrow[dd, "s_\kappa"] \\
& & & X^*_\omega \arrow[ur, "e^*_\omega"] \arrow[dr, "s^*_\omega"'] \arrow[r, phantom, "\mathsf{po}"] \arrow[d, phantom, "\mathsf{colim}"] & {} & & & & \\
X \arrow[r, "e_{0,1}", "\mathsf{alg\ coeq}"'] & X_1 \arrow[rrr, "e^\circ_{1,\omega}"'] \arrow[rrrrr, bend right, "e_{1,\omega}"'] & \cdots\cdots & {} & X^\circ_\omega \arrow[r, "e^\circ_\omega"] \arrow[d, phantom, "\mathsf{colim}"] & X^\bullet_\omega \arrow[r, "e^\bullet_\omega"] & X_\omega & \cdots & X_\kappa \\
\Gamma X \arrow[u, shift left, "L(s)^\diamond"] \arrow[u, shift right, "R(s)^\diamond"'] \arrow[r, "\Gamma e_{0,1}"] & \Gamma X_1 \arrow[u, shift left, "L(s_1)^\diamond"] \arrow[u, shift right, "R(s_1)^\diamond"'] \arrow[rrrrr, bend right, "\Gamma e_{1,\omega}"'] & & \cdots\cdots & {} & & \Gamma X_\omega \arrow[u, shift left, "L(s_\omega)^\diamond"] \arrow[u, shift right, "R(s_\omega)^\diamond"'] & \cdots & \Gamma X_\kappa \arrow[u, shift left, "L(s_\kappa)^\diamond"] \arrow[u, shift right, "R(s_\kappa)^\diamond"']
\end{tikzcd}
\qquad (**)
$$

yielding a chain $\{\, e_{\alpha,\beta} : (X_\alpha, s_\alpha) \to (X_\beta, s_\beta) \,\}_{\alpha \le \beta \le \kappa}$ (with $s_0 = s$) in Σ-**Alg**.

Precisely, the definitions are as follows: for $\lambda \le \kappa$,

- when $\lambda = 0$,
  (X_λ, s_λ) is (X, s);

- when λ is a successor ordinal $\alpha + 1$,
  $e_{\alpha,\lambda} : (X_\alpha, s_\alpha) \to (X_\lambda, s_\lambda)$ is a Σ-algebraic coequalizer of $L(X_\alpha, s_\alpha)^\diamond, R(X_\alpha, s_\alpha)^\diamond : \Gamma X_\alpha \to X_\alpha$; and

- when λ is a limit ordinal,
  - $\{\, e^\circ_{\alpha,\lambda} : X_\alpha \to X^\circ_\lambda \,\}_{\alpha<\lambda}$ and $\{\, e^*_{\alpha,\lambda} : \Sigma X_\alpha \to X^*_\lambda \,\}_{\alpha<\lambda}$ are respectively colimits of the λ-chains $\{\, e_{\alpha,\beta} \,\}_{\alpha\le\beta<\lambda}$ and $\{\, \Sigma e_{\alpha,\beta} \,\}_{\alpha\le\beta<\lambda}$;
  - $e^*_\lambda : X^*_\lambda \to \Sigma X^\circ_\lambda$ and $s^*_\lambda : X^*_\lambda \to X^\circ_\lambda$ are the mediating maps from the colimiting cone $\{\, e^*_{\alpha,\lambda} \,\}_{\alpha<\lambda}$ to the cones $\{\, \Sigma e^\circ_{\alpha,\lambda} \,\}_{\alpha<\lambda}$ and $\{\, e^\circ_{\alpha,\lambda} \circ s_\alpha \,\}_{\alpha<\lambda}$;
  - $(X^\circ_\lambda \xrightarrow{e^\circ_\lambda} X^\bullet_\lambda \xleftarrow{s^\circ_\lambda} \Sigma X^\circ_\lambda)$ is a pushout of $(X^\circ_\lambda \xleftarrow{s^*_\lambda} X^*_\lambda \xrightarrow{e^*_\lambda} \Sigma X^\circ_\lambda)$;
  - (X_λ, s_λ) is a free Σ-algebra on the Σ-algebra cospan $(X^\circ_\lambda \xrightarrow{e^\circ_\lambda} X^\bullet_\lambda \xleftarrow{s^\circ_\lambda} \Sigma X^\circ_\lambda)$ with universal map $(e^\bullet_\lambda \circ e^\circ_\lambda, e^\bullet_\lambda)$; and
  - $e_{\alpha,\lambda} : X_\alpha \to X_\lambda$ is the composite $e^\bullet_\lambda \circ e^\circ_\lambda \circ e^\circ_{\alpha,\lambda}$ for $\alpha < \lambda$.

Definition 4.1.8. Whenever the construction $(**)$ above can be performed for the ordinal κ, we say that it *reaches* κ. Furthermore, we say that the construction $(**)$ *stops* if it does so at some ordinal κ in the sense that it reaches $\kappa + 1$ and the map $e_{\kappa,\kappa+1} : X_\kappa \to X_{\kappa+1}$ is an isomorphism.

We now show that if the construction (∗∗) stops, it constructs a free $\mathbb{S}$-algebra on the Σ-algebra (X, s).

Theorem 4.1.9. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system. If the construction (∗∗) stops for all Σ-algebras, then $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**.*

Proof. Let (X, s) be a Σ-algebra and assume that the construction (∗∗) for it stops at an ordinal κ. We claim that the Σ-algebra (X_κ, s_κ) is a free $\mathbb{S}$-algebra on (X, s).

First, note that (X_κ, s_κ) is an $\mathbb{S}$-algebra since $e_{\kappa,\kappa+1} \circ L(X_\kappa, s_\kappa)^\diamond = e_{\kappa,\kappa+1} \circ R(X_\kappa, s_\kappa)^\diamond$ and $e_{\kappa,\kappa+1}$ is an isomorphism.

Second, we show that the homomorphism $e_{0,\kappa} : (X, s) \to (X_\kappa, s_\kappa)$ is universal. To this end, consider a homomorphism $h : (X, s) \to (W, u)$ and perform the following construction:

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma e_{0,1}"] \arrow[dd, "s"] \arrow[drrrrrrrr, "\Sigma h"] & \Sigma X_1 \arrow[rr, bend left, "\Sigma e^\circ_{1,\omega}", "\cdots\cdots"'] \arrow[dd, "s_1"] \arrow[dr, "e^*_{1,\omega}"'] \arrow[drrrrrrr, "\Sigma h_1"] & & \Sigma X^\circ_\omega \arrow[rr, "\Sigma(e^\bullet_\omega \circ e^\circ_\omega)"] \arrow[ddr, "s^\circ_\omega"] \arrow[drrrrr, "\Sigma h^\circ_\omega"] & & \Sigma X_\omega \arrow[dd, "s_\omega"] \arrow[drrr, "\Sigma h_\omega"] & \cdots & \Sigma X_\kappa \arrow[dd, "s_\kappa"] \arrow[dr, "\Sigma h_\kappa"] & \\
& & X^*_\omega \arrow[ur, "e^*_\omega"] \arrow[dr, "s^*_\omega"] & & & & & & \Sigma W \arrow[dd, "u"] \\
X \arrow[r, "e_{0,1}"] \arrow[drrrrrrrr, "h"] & X_1 \arrow[rr, bend left, "e^\circ_{1,\omega}"', "\cdots\cdots"] \arrow[drrrrrrr, "h_1"] & & X^\circ_\omega \arrow[r, "e^\circ_\omega"] \arrow[drrrrr, "h^\circ_\omega"] & X^\bullet_\omega \arrow[r, "e^\bullet_\omega"] \arrow[drrrr, "h^\bullet_\omega"] & X_\omega \arrow[drrr, "h_\omega"] & \cdots & X_\kappa \arrow[dr, "h_\kappa"] & \\
& & & & & & & & W \\
\Gamma X \arrow[r, "\Gamma e_{0,1}"] \arrow[uu, shift left, "L(s)^\diamond"] \arrow[uu, shift right, "R(s)^\diamond"'] \arrow[drrrrrrrr, "\Gamma h"'] & \Gamma X_1 \arrow[rrrr, bend left, "\Gamma e_{1,\omega}", "\cdots\cdots"'] \arrow[uu, shift left, "L(s_1)^\diamond"] \arrow[uu, shift right, "R(s_1)^\diamond"'] \arrow[drrrrrrr, "\Gamma h_1"] & & & & \Gamma X_\omega \arrow[uu, shift left, "L(s_\omega)^\diamond"] \arrow[uu, shift right, "R(s_\omega)^\diamond"'] \arrow[drrr, "\Gamma h_\omega"] & \cdots & \Gamma X_\kappa \arrow[uu, "L(s_\kappa)^\diamond = R(s_\kappa)^\diamond"] \arrow[dr, "\Gamma h_\kappa"] & \\
& & & & & & & & \Gamma W \arrow[uu, "L(u)^\diamond = R(u)^\diamond"]
\end{tikzcd}$$

where

- for $\lambda = 0$, h_λ is h;
- for a successor ordinal $\lambda = \alpha + 1$,
  $h_\lambda : (X_\lambda, s_\lambda) \to (W, u)$ is the factor of h_α through the algebraic coequalizer $e_{\alpha,\alpha+1}$; and
- for a limit ordinal λ,
  - h°_λ is the mediating map from the colimit X°_λ to W with respect to the cone $\{\, h_\alpha \,\}_{\alpha<\lambda}$;
  - $h^\bullet_\lambda$ is the mediating map from the pushout $X^\bullet_\lambda$ to W with respect to the cone $(h^\circ_\lambda : X^\circ_\lambda \to W \leftarrow \Sigma X^\circ_\lambda : u \circ \Sigma h^\circ_\lambda)$; and
  - $h_\lambda : (X_\lambda, s_\lambda) \to (W, u)$ is the factor of
  $$(h^\circ_\lambda, h^\bullet_\lambda) : (X^\circ_\lambda \xrightarrow{e^\circ_\lambda} X^\bullet_\lambda \xleftarrow{s^\circ_\lambda} \Sigma X^\circ_\lambda) \to (W \xrightarrow{\mathrm{id}} W \xleftarrow{u} \Sigma W)$$
  through the universal map $(e^\bullet_\lambda \circ e^\circ_\lambda, e^\bullet_\lambda)$.

It thus follows that $h_\kappa : (X_\kappa, s_\kappa) \to (W, u)$ is a factor of $h : (X, s) \to (W, u)$ through $e_{0,\kappa} : (X, s) \to (X_\kappa, s_\kappa)$.

We finally establish the uniqueness of such factorizations. Indeed, for any homomorphism $g : (X_\kappa, s_\kappa) \to (W, u)$ such that $g \circ e_{0,\kappa} = h$, it follows by transfinite induction that $g \circ e_{\alpha,\kappa} = h_\alpha$ for all $\alpha \leq \kappa$, and hence that $g = h_\kappa$. □

4.1.4 Sufficient conditions for free constructions

We conclude the section by giving sufficient conditions that permit the application of Corollary 4.1.7 and Theorem 4.1.9, and thus lead to transfinite constructions of free algebras for equational systems.

Theorem 4.1.10. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system with $\mathscr{C}$ finitely and chain cocomplete.*

1. *If Σ is κ-cocontinuous for some infinite limit ordinal κ, then the construction $(*)$ stops at κ for all Σ-algebra cospans.*

2. *In addition, if Γ is κ-cocontinuous, or if both Σ and Γ are epicontinuous, then the construction $(**)$ respectively stops at κ, or at 1, for every Σ-algebra.*

Proof. Assume that Σ is κ-cocontinuous for some infinite limit ordinal κ. As $\mathscr{C}$ is finitely and chain cocomplete, the construction $(*)$ for a Σ-algebra cospan $(Z \xrightarrow{c} Z' \xleftarrow{t} \Sigma Z)$ reaches the ordinal κ. As the functor Σ preserves the colimiting cone $\{ c_{\alpha,\kappa} \}_{\alpha<\kappa}$ of the κ-chain $\{ c_{\alpha,\beta} \}_{\alpha\leq\beta<\kappa}$, the mediating map c^*_κ is an isomorphism and hence so is $c_{\kappa,\kappa+1}$.

By Theorem 4.1.5, free Σ-algebras on Σ-algebra cospans exist; and so do Σ-algebraic coequalizers by Corollary 4.1.6. Thus, the construction $(**)$ reaches every ordinal.

In addition, assume that Γ is κ-cocontinuous, and consider the construction $(**)$ for a Σ-algebra (X, s) up to the ordinal $\kappa + 1$. As Σ preserves the colimiting cone $\{ e^\circ_{\alpha,\kappa} \}_{\alpha<\kappa}$ of the κ-chain $\{ e_{\alpha,\beta} \}_{\alpha\leq\beta<\kappa}$, the mediating map e^*_κ is an isomorphism and hence so are e°_κ and $e^\bullet_\kappa$. From this, we have that $\{ e_{\alpha,\kappa} \}_{\alpha<\kappa}$ is a colimiting cone of the κ-chain $\{ e_{\alpha,\beta} \}_{\alpha\leq\beta<\kappa}$. Since Γ preserves it and the equation $L(X_\kappa, s_\kappa)^\diamond \circ \Gamma e_{\alpha,\kappa} = R(X_\kappa, s_\kappa)^\diamond \circ \Gamma e_{\alpha,\kappa}$ holds for every $\alpha < \kappa$, it follows that $L(X_\kappa, s_\kappa)^\diamond = R(X_\kappa, s_\kappa)^\diamond$. Therefore, the algebraic coequalizer $e_{\kappa,\kappa+1}$ is an isomorphism.

Alternatively, besides κ-cocontinuity of Σ, assume both that Σ and Γ are epicontinuous, and consider the construction $(**)$ for a Σ-algebra (X, s) up to the ordinal 2. Then, by Corollary 4.1.6, the Σ-algebraic coequalizer $e_{0,1}$ is epimorphic in $\mathscr{C}$, and thus so is $\Gamma e_{0,1}$. Since $\Gamma e_{0,1}$ equalizes $L(X_1, s_1)^\diamond$ and $R(X_1, s_1)^\diamond$, we have that $L(X_1, s_1)^\diamond = R(X_1, s_1)^\diamond$. Therefore, the algebraic coequalizer $e_{1,2}$ is an isomorphism. □

Theorem 4.1.11. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system with $\mathscr{C}$ finitely and chain cocomplete. If $\mathscr{C}$ has no transfinite chain of proper epimorphisms and Σ is epi-*

continuous, then the construction (*) stops for all Σ-algebra cospans $(c : Z \to Z' \leftarrow \Sigma Z : t)$ with c epimorphic, and the construction (**) stops for all Σ-algebras.

Proof. As $\mathscr{C}$ is a finitely and chain cocomplete category, the construction $(*)$ for a Σ-algebra cospan $(c : Z \to Z' \leftarrow \Sigma Z : t)$ with c epimorphic reaches every ordinal. As Σ is epicontinuous, it follows by transfinite induction that the maps $c_{\alpha,\beta}$ and $\Sigma c_{\alpha,\beta}$ are epimorphic for all ordinals $\alpha \leq \beta$. Since $\{ c_{\alpha,\beta} \}_{\alpha\leq\beta\in\mathbf{Ord}}$ is a transfinite chain of epimorphisms, there exists, by hypothesis, an isomorphic component $c_{\alpha,\beta}$ for some pair of ordinals $\alpha < \beta$. Thus the construction stops.

From Theorem 4.1.5 and Corollary 4.1.6, it follows that free Σ-algebras on Σ-algebra cospans $(c : Z \to Z' \leftarrow \Sigma Z : t)$ with c epimorphic and Σ-algebraic coequalizers exist, and that their associated universal maps are epimorphic in $\mathscr{C}$. Thus, the construction $(**)$ reaches every ordinal for all Σ-algebras; and, by transfinite induction, the maps $e_{\alpha,\beta}$, $e^{\circ}_{\gamma,\lambda}$, $e^{*}_{\gamma,\lambda}$, e^{*}_{λ}, e°_{λ}, $e^{\bullet}_{\lambda}$ are shown to be epimorphic in $\mathscr{C}$, for all $\alpha \leq \beta \in \mathbf{Ord}$ and $\gamma < \lambda \in \mathbf{Ord}$ with λ a limit ordinal. Since $\{ e_{\alpha,\beta} \}_{\alpha\leq\beta\in\mathbf{Ord}}$ is a transfinite chain of epimorphisms, there exists, by hypothesis, an isomorphic component $e_{\alpha,\beta}$ for some pair of ordinals $\alpha < \beta$. Thus the construction stops. □

The following two corollaries imply Theorems 4.1.1 and 4.1.2.

Corollary 4.1.12. *Let Σ be an endofunctor on a category $\mathscr{C}$. For $\mathscr{C}$ finitely and chain cocomplete, if Σ is κ-cocontinuous for some infinite limit ordinal κ, then the forgetful functor Σ-$\mathbf{Alg} \to \mathscr{C}$ has a left adjoint.*

Proof. From Corollary 4.1.7 and Theorem 4.1.10 (as the endofunctors $X + \Sigma(-)$ on $\mathscr{C}$ is κ-cocontinuous for all $X \in \mathscr{C}$), it follows that a free Σ-algebra on an object in $\mathscr{C}$ is constructed as in the diagram (4.1). □

Corollary 4.1.13. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system. For $\mathscr{C}$ finitely and chain cocomplete, if either of the following conditions hold*

1. *Σ and Γ are κ-cocontinuous for some infinite limit ordinal κ;*

2. *Σ is κ-cocontinuous for some infinite limit ordinal κ, and both Σ and Γ are epicontinuous;*

3. *$\mathscr{C}$ has no transfinite chain of proper epimorphisms and Σ is epicontinuous*

then $\mathbb{S}$-$\mathbf{Alg}$ is a full reflective subcategory of Σ-$\mathbf{Alg}$.

Proof. Items 1 and 2 follow from Theorems 4.1.9 and 4.1.10; item 3 follows from Theorems 4.1.9 and 4.1.11. □

Note that under the condition of item 2 above, the construction of a free $\mathbb{S}$-algebra over a Σ-algebra (X, s) is carried out in κ steps as described below:

$$
\begin{tikzcd}
& \Sigma X \arrow[r, two heads, "\Sigma c"] \arrow[dd, "s"'] \arrow[ddr, "c \circ s"] \arrow[ddrr, phantom, "\mathsf{po}"] & \Sigma X_1 \arrow[r, two heads, "\Sigma c_{1,2}"] \arrow[ddr, "s_1"] & \Sigma X_2 \arrow[rr, two heads, bend left, "\Sigma c_{2,\omega}", "\cdots\cdots"'] \arrow[dr, "\ddots", "c^*_{2,\omega}"'] & & \Sigma X_\omega \arrow[r, two heads, "\Sigma c_{\omega,\omega+1}"] \arrow[ddr, "t_\omega"] \arrow[dd, phantom, "\mathsf{po}"] & \Sigma X_{\omega+1} & \cdots & \Sigma X_\kappa \arrow[d, "t_\kappa"] \\
& & & & X^*_\omega \arrow[ur, "c^*_\omega"'] \arrow[dr, "t^*_\omega"] \arrow[d, phantom, "\mathsf{colim}"] & & & & X_{\kappa+1} \arrow[d, "c^{-1}_{\kappa,\kappa+1}"', "\cong"] \\
\Gamma X \arrow[r, shift left, "L(X,s)^\diamond"] \arrow[r, shift right, "R(X,s)^\diamond"'] & X \arrow[r, two heads, "c", "\mathsf{coeq}"'] & X_1 \arrow[r, two heads, "c_{1,2}"'] & X_2 \arrow[rr, two heads, bend right, "\cdots\cdots", "c_{2,\omega}"'] & & X_\omega \arrow[r, two heads, "c_{\omega,\omega+1}"'] \arrow[d, phantom, "\mathsf{colim}"] & X_{\omega+1} & \cdots & X_\kappa \\
& & & & & & & &
\end{tikzcd}
\tag{4.2}
$$

where the map $c : X \to X_1$ is a coequalizer of the parallel pair $L(X, s)^\diamond, R(X, s)^\diamond$.

4.1.5 Construction of free algebras for monadic equational systems

We briefly discuss free constructions for finitary and inductive monadic equational systems.

Definition 4.1.14. A monadic equational system $\mathbb{S} = (\mathscr{C} : \mathbf{T} \rhd \Gamma \vdash L \equiv R)$ with $\mathbf{T} = (T, \eta, \mu)$ is said to be *κ-finitary*, for κ an infinite limit ordinal, if the category $\mathscr{C}$ is cocomplete and both functors T and Γ are κ-cocontinuous. Such a monadic equational system is said to be *κ-inductive* if furthermore both functors T and Γ are epicontinuous.

One can construct free algebras for κ-finitary monadic equational systems, for any infinite limit ordinal κ, via the encoding of these systems into κ-finitary (ordinary) equational systems given in Section 2.3. However, when the systems are κ-inductive, the construction can be simplified (see Corollary 4.1.16 below).

Theorem 4.1.15. *Let κ be an infinite limit ordinal and $\mathbb{S} = (\mathscr{C} : \mathbf{T} \rhd \Gamma \vdash L \equiv R)$ a κ-inductive monadic equational system with $\mathbf{T} = (T, \eta, \mu)$. For an Eilenberg-Moore algebra (X, s) of the monad $\mathbf{T}$, every T-algebraic coequalizer of $L(X, s)^\diamond$, $R(X, s)^\diamond$ into the T-algebra (X, s) yields a free $\mathbb{S}$-algebra $(\widetilde{X}, \widetilde{s})$ over (X, s). Hence, this construction provides a left adjoint to $J : \mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow \mathscr{C}^{\mathbf{T}}$.*

$$
\begin{tikzcd}
& TX \arrow[r, two heads, "Tq"] \arrow[d, "s"] & T\widetilde{X} \arrow[d, "\widetilde{s}"] \\
\Gamma X \arrow[r, shift left, "L(X,s)^\diamond"] \arrow[r, shift right, "R(X,s)^\diamond"'] & X \arrow[r, two heads, "q", "\mathsf{algcoeq}"'] & \widetilde{X}
\end{tikzcd}
$$

Proof. This theorem holds by the same argument as in Theorem 3.2.11; because, for the equivalent κ-inductive equational system $\overline{\mathbb{S}}$ with functorial signature T given in Section 2.3, a free $\overline{\mathbb{S}}$-algebra over a T-algebra is constructed by means of a T-algebraic coequalizer (see Theorems 4.1.9 and Theorem 4.1.10 (2)). $\square$

For each object X in $\mathscr{C}$, as $(TX, \mu_X : TTX \to TX)$ is a free Eilenberg-Moore algebra on X, we have the following corollary.

Corollary 4.1.16. *Let κ be an infinite limit ordinal and $\mathbb{S} = (\mathscr{C} : \mathbf{T} \rhd \Gamma \vdash L \equiv R)$ a κ-inductive monadic equational system with $\mathbf{T} = (T, \eta, \mu)$. For each object X in $\mathscr{C}$, every T-algebraic coequalizer of $L(TX, \mu_X)^\diamond$, $R(TX, \mu_X)^\diamond$ into (TX, μ_X) yields a free $\mathbb{S}$-algebra on X.*

4.2 Properties of equational systems

We consider properties of categories of algebras and representing monads for equational systems. Overall, the following two theorems are established: the first one by Corollaries 4.2.4, 4.2.7, and 4.2.10; the second one by Theorem 4.1.2 and Corollary 4.2.5.

Theorem 4.2.1. *For a κ-finitary equational system $\mathbb{S}$ on a category $\mathscr{C}$, the category $\mathbb{S}$-**Alg** is cocomplete, the forgetful functor $U_\mathbb{S} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic, and the representing monad of $\mathbb{S}$ is κ-cocontinuous. Furthermore, if the system $\mathbb{S}$ is κ-inductive, the representing monad is also epicontinuous.*

Theorem 4.2.2. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system with $\mathscr{C}$ cocomplete. If $\mathscr{C}$ has no transfinite chain of proper epimorphisms, and Σ is epicontinuous and κ-cocontinuous for some infinite limit ordinal κ, then $U_\mathbb{S} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic and $\mathbb{S}$-**Alg** is cocomplete.*

4.2.1 Properties of categories of algebras

General conditions for the monadicity and cocompleteness of categories of algebras for equational systems are given.

Proposition 4.2.3. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system with $\mathscr{C}$ cocomplete. If the forgetful functor $U_\mathbb{S} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**, and Σ-**Alg** has coequalizers, then $\mathbb{S}$-**Alg** is cocomplete.*

Proof. The category $\mathbb{S}$-**Alg** has coequalizers since it is a full reflective subcategory of Σ-**Alg**, which is assumed to have coequalizers. Also, by Proposition 3.3.4, $\mathbb{S}$-**Alg** is monadic over $\mathscr{C}$. Being monadic over a cocomplete category and having coequalizers, $\mathbb{S}$-**Alg** is cocomplete (see *e.g.* [Borceux 1994, Proposition 4.3.4]). □

Recalling from Lemma 3.3.5 that the existence of Σ-algebraic coequalizers implies that of coequalizers in Σ-**Alg**, we obtain the following corollaries.

Corollary 4.2.4. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system. For $\mathscr{C}$ cocomplete, if either of the following conditions hold*

1. *Σ and Γ are κ-cocontinuous for some infinite limit ordinal κ;*

2. *Σ is κ-cocontinuous for some infinite limit ordinal κ, and both Σ and Γ are epicontinuous*

then the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic and $\mathbb{S}\text{-}\mathbf{Alg}$ is cocomplete.

Proof. By Proposition 3.3.4 and Corollaries 4.1.12 and 4.1.13, it follows that the forgetful functor $U_{\mathbb{S}}$ is monadic; by Corollary 4.1.13, that $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**; and, by Lemma 3.3.5, Corollary 4.1.6, and Theorem 4.1.10, that Σ-**Alg** has coequalizers. Thus, it follows from Proposition 4.2.3 that $\mathbb{S}$-**Alg** is cocomplete. $\square$

Corollary 4.2.5. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system with $\mathscr{C}$ cocomplete such that the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint. If $\mathscr{C}$ has no transfinite chain of proper epimorphisms, and Σ is epicontinuous, then the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic and $\mathbb{S}\text{-}\mathbf{Alg}$ is cocomplete.*

Proof. By Proposition 3.3.4, it follows that the forgetful functor $U_{\mathbb{S}}$ is monadic; by Corollary 4.1.13, that $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**; and, by Lemma 3.3.5, Corollary 4.1.6, and Theorem 4.1.11, that Σ-**Alg** has coequalizers. Thus, it follows from Proposition 4.2.3 that $\mathbb{S}$-**Alg** is cocomplete. $\square$

Remark 4.2.6. Note that in Corollary 4.2.5 above $U_{\mathbb{S}}$ having a left adjoint is a weaker condition than U_Σ doing so, as $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**. Indeed, for the usual powerset monad $(\mathscr{P}, \{\cdot\}, \cup)$ on **Set**, the endofunctor $\mathscr{P}$ is epicontinuous, the category **Set** has no transfinite chain of proper epimorphisms, and the forgetful functor from the category $\mathbf{Set}^{\mathscr{P}}$ of Eilenberg-Moore algebras has a left adjoint, but that from the category $\mathscr{P}$-**Alg** of $\mathscr{P}$-algebras has no left adjoint due to a size problem.

4.2.2 Properties of representing monads

The cocontinuity of representing monads of finitary equational systems directly follows from Proposition 3.3.7.

Corollary 4.2.7. *For any infinite limit ordinal κ, the representing monads of κ-finitary equational systems are κ-cocontinuous.*

The epicontinuity of representing monads of inductive equational systems follows from the following two propositions.

Proposition 4.2.8. *Let Σ be an endofunctor on a category $\mathscr{C}$. If $\mathscr{C}$ is finitely and chain cocomplete and Σ is epicontinuous and κ-cocontinuous for some infinite limit ordinal κ, then the monad T_Σ induced by the left adjoint to the forgetful functor $U_\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ is epicontinuous.*

Proof. Recall from Theorem 4.1.10 and Corollary 4.1.7 that the construction (4.1) stops at κ and yields an initial $(X + \Sigma(-))$-algebra $(X_\kappa, [\eta_X, \tau_X] : X + \Sigma X_\kappa \xrightarrow{\cong} X_\kappa)$ whose component $\tau_X : \Sigma X_\kappa \to X_\kappa$ is a free Σ-algebra on X. The monad T_Σ is induced from this free construction and given as follows:

- $T_\Sigma X = X_\kappa$ for each object $X \in \mathscr{C}$, and
- $T_\Sigma f$ for each morphism $f : X \to Y$ in $\mathscr{C}$ is the unique morphism making the following diagram commutative.

$$\begin{tikzcd}
\Sigma X_\kappa \arrow[r, "\Sigma(T_\Sigma f)"] \arrow[d, "\tau_X"'] & \Sigma Y_\kappa \arrow[d, "\tau_Y"] \\
X_\kappa \arrow[r, "\exists!\, T_\Sigma f"] & Y_\kappa \\
X \arrow[u, "\eta_X"] \arrow[r, "f"] & Y \arrow[u, "\eta_Y"']
\end{tikzcd}$$

Given an epimorphism $f : X \twoheadrightarrow Y$ in $\mathscr{C}$, one can construct a family of epimorphisms $\{ f_\alpha : X_\alpha \twoheadrightarrow Y_\alpha \}_{\alpha \le \kappa+1}$ such that

$$\begin{tikzcd}
X_\alpha \arrow[r] \arrow[d, two heads, "f_\alpha"'] & X_\beta \arrow[d, two heads, "f_\beta"] \\
Y_\alpha \arrow[r] & Y_\beta
\end{tikzcd}$$

commutes by setting

- $f_0 = \mathrm{id}$;
- $f_{\alpha+1} = f + \Sigma f_\alpha$ for all successor ordinals $\alpha + 1$; and
- f_λ as the unique mediating map from the colimiting cone $\{ X_\alpha \to X_\lambda \}_{\alpha<\lambda}$ to the cone $\{ X_\alpha \xrightarrow{f_\alpha} Y_\alpha \to Y_\lambda \}_{\alpha<\lambda}$ for all limit ordinals λ.

$$\begin{tikzcd}[column sep=small]
0 \arrow[r, "!"] \arrow[d, two heads, "0"] & X + \Sigma 0 \arrow[r, "X+\Sigma !"] \arrow[d, two heads, "f+\Sigma 0"] & X + \Sigma(X + \Sigma 0) \arrow[d, two heads, "f+\Sigma(f+\Sigma 0)"] & \cdots & X_\omega \arrow[r] \arrow[d, two heads, "f_\omega"] & X + \Sigma X_\omega \arrow[d, two heads, "f+\Sigma f_\omega"] & \cdots & X_\kappa \arrow[r, "{[\eta_X, \tau_X]^{-1}}", "\cong"'] \arrow[d, two heads, "f_\kappa"] \arrow[dr, phantom, "\mathsf{(A)}"] & X + \Sigma X_\kappa \arrow[d, two heads, "f+\Sigma(f_\kappa)"] \\
0 \arrow[r, "!"] & Y + \Sigma 0 \arrow[r, "Y+\Sigma !"] & Y + \Sigma(Y + \Sigma 0) & \cdots & \underset{\mathsf{colim}}{Y_\omega} \arrow[r] & Y + \Sigma Y_\omega & \cdots & Y_\kappa \arrow[r, "{[\eta_Y, \tau_Y]^{-1}}", "\cong"'] & Y + \Sigma Y_\kappa
\end{tikzcd}$$

From the commutativity of the subdiagram (A), it follows that $f_\kappa \circ \eta_X = \eta_Y \circ f$ and that $f_\kappa \circ \tau_X = \tau_Y \circ \Sigma(f_\kappa)$. Thus, $T_\Sigma f = f_\kappa$ is an epimorphism. □

Proposition 4.2.9. *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ be an equational system with $\mathscr{C}$ finitely and chain cocomplete. If Σ is both epicontinuous and κ-cocontinuous for some infinite limit ordinal κ, and if Γ is either epicontinuous or κ-cocontinuous, then the representing monad $T_\mathbb{S}$ of $\mathbb{S}$ is epicontinuous.*

Proof. By Corollaries 4.1.12 and 4.1.13, for $X \in \mathscr{C}$, the free $\mathbb{S}$-algebra $(T_{\mathbb{S}}X, \widetilde{\tau}_X)$ over the free Σ-algebra $(T_\Sigma X, \tau_X)$ on X is given by means of the constructions $(*)$ and $(**)$. As Σ is epicontinuous, it follows that the universal homomorphism $q_X : (T_\Sigma X, \tau_X) \to (T_{\mathbb{S}}X, \widetilde{\tau}_X)$ is epimorphic in $\mathscr{C}$. Thus, by Proposition 4.2.8, for every epimorphism $f : X \twoheadrightarrow Y$, we have the following situation

$$\begin{tikzcd} T_\Sigma X \arrow[r, two heads, "q_X"] \arrow[d, two heads, "T_\Sigma f"'] \arrow[dr, two heads] & T_{\mathbb{S}}X \arrow[d, "T_{\mathbb{S}}f"] \\ T_\Sigma Y \arrow[r, two heads, "q_Y"'] & T_{\mathbb{S}}Y \end{tikzcd}$$

and, as $T_{\mathbb{S}}f \circ q_X = q_Y \circ T_\Sigma f$ is an epimorphism, so is $T_{\mathbb{S}}f$. □

Corollary 4.2.10. *For any infinite limit ordinal κ, the representing monads of κ-inductive equational systems are epicontinuous.*

4.2.3 Properties of categories of finitary and inductive equational systems

The cocompleteness of the categories of finitary and of inductive equational systems on a cocomplete category follows as a corollary of Proposition 3.3.10.

Corollary 4.2.11. *For a cocomplete category $\mathscr{C}$ and an infinite limit ordinal κ, the full subcategories κ-**FinES**$(\mathscr{C})$ and κ-**IndES**$(\mathscr{C})$ of **ES**$(\mathscr{C})$ respectively consisting of κ-finitary equational systems and of κ-inductive equational systems are cocomplete.*

Proof. As every κ-finitary (resp. κ-inductive) equational system has a representing monad, it follows from Proposition 3.3.10 that small coproducts and coequalizers of κ-finitary (resp. κ-inductive) equational systems exist in **ES**$(\mathscr{C})$. Moreover, that those colimits in **ES**$(\mathscr{C})$ are also κ-finitary (resp. κ-inductive) follows from (3.5) and (3.6), and from the fact that representing monads of κ-finitary (resp. κ-inductive) equational systems are κ-cocontinuous (and epicontinuous), by Corollary 4.2.7 (and Corollary 4.2.10). Thus, the categories κ-**FinES**$(\mathscr{C})$ and κ-**IndES**$(\mathscr{C})$ have small colimits. □

4.3 Examples

We revisit the examples of equational systems given in Section 2.5 in the light of the results of this chapter.

1. See item 1 of Section 3.4 (as every algebraic theory induces an ω-inductive equational system).

2. For the equational system $\mathbb{S}_{\mathbb{T}} = (\mathscr{C}_0 : \Sigma_{\mathbb{T}} \rhd \Gamma_{\mathbb{T}} \vdash L_{\mathbb{T}} \equiv R_{\mathbb{T}})$ representing an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, \Sigma, E)$, as the system $\mathbb{S}_{\mathbb{T}}$ is ω-finitary, we can apply

Theorem 4.2.1 as follows: the category $\mathbb{S}_{\mathbb{T}}$-**Alg** is monadic over $\mathscr{C}_0$ and cocomplete; and the free-algebra monad (*i.e.*, the representing monad of $\mathbb{S}_{\mathbb{T}}$) is ω-cocontinuous. By Proposition 3.3.7, the free-algebra monad can be further shown to be finitary (*i.e.*, it preserves filtered colimits), as the endofunctors $\Sigma_{\mathbb{T}}$ and $\Gamma_{\mathbb{T}}$ are finitary.

3. One may apply Theorem 4.2.1 to the equational system $\mathbb{S}_{\mathbf{T}}$ representing a monad $\mathbf{T} = (T, \eta, \mu)$ on a cocomplete category $\mathscr{C}$ as follows. If T is κ-cocontinuous for some infinite limit ordinal κ, then $\mathbb{S}_{\mathbf{T}}$-**Alg** $\cong \mathscr{C}^{\mathbf{T}}$ is cocomplete.

   One may also apply Corollary 4.2.5 as follows. If $\mathscr{C}$ has no transfinite chain of proper epimorphisms and T is epicontinuous, then $\mathbb{S}_{\mathbf{T}}$-**Alg** $\cong \mathscr{C}^{\mathbf{T}}$ is cocomplete.

   Moreover, for an infinite limit ordinal κ, the category of κ-cocontinuous monads on a cocomplete category $\mathscr{C}$ is equivalent to the category κ-**FinES**$(\mathscr{C})$ through

   a) the embedding sending a monad $\mathbf{T}$ to the equational system $\mathbb{S}_{\mathbf{T}}$ (see item 3 of Section 2.5), and

   b) the embedding sending a κ-finitary equational system to its representing monad.

   Thus, by Corollary 4.2.11, it follows that the category of κ-cocontinuous monads on $\mathscr{C}$ is cocomplete.

   Similarly, one can show that the category **FinMnd**$(\mathscr{C})$ of finitary monads (*i.e.*, those that preserve filtered colimits) on a cocomplete category $\mathscr{C}$ is equivalent to the full subcategory of **ES**$(\mathscr{C})$ consisting of equational systems whose functorial signatures and contexts preserve filtered colimits. As every functor preserving colimits of κ-chains for all infinite ordinal κ is finitary, and vice versa, the above full subcategory is the intersection of the full subcategories κ-**FinES**$(\mathscr{C})$ in **ES**$(\mathscr{C})$ for all infinite limit ordinals κ. Since, by Corollary 4.2.11, the full subcategories κ-**FinES**$(\mathscr{C})$ of **ES**$(\mathscr{C})$ are closed under the colimits given by Proposition 3.3.10, it follows that the category **FinMnd**$(\mathscr{C})$ is cocomplete.

4. To the equational system $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$ of monoids in a cocomplete monoidal category $\mathscr{C}$, we can apply Theorem 4.2.1 as follows. If the tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is κ-cocontinuous for some infinite limit ordinal κ (as it happens, for instance, when it is biclosed) then $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$-**Alg** (*i.e.*, the category of monoids in $\mathscr{C}$) is cocomplete and monadic over $\mathscr{C}$, and the free-monoid monad (*i.e.*, the representing monad of $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$) is κ-cocontinuous. If, in addition, the tensor product is epicontinuous (again, as it happens when it is biclosed), then so does the free-monoid monad. Furthermore, by Proposition 3.3.7, if the tensor product is finitary (also as it happens when it is biclosed) then the free-monoid monad is finitary.

5. To the equational cosystem $\mathbb{S}_{\mathrm{CoMon}(\mathscr{C})}$ of comonoids in a complete monoidal category $\mathscr{C}$, we may apply the dual version of Theorem 4.2.1 as follows. If the tensor

product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ preserves limits of κ-cochains for some infinite limit ordinal κ then $\mathbb{S}_{\mathrm{CoMon}(\mathscr{C})}$-**CoAlg** (*i.e.*, the category of comonoids in $\mathscr{C}$) is complete and comonadic over $\mathscr{C}$, and the cofree-comonoid comonad (*i.e.*, the representing comonad of $\mathbb{S}_{\mathrm{CoMon}(\mathscr{C})}$) preserves limits of κ-cochains. If, in addition, the tensor product preserves monomorphisms, then so does the free-monoid monad.

Chapter 5

Applications

For the two modern applications, π-*algebras* [Stark 2005, 2008] and Σ-*monoids* [Fiore et al. 1999], we discuss

- difficulties in representing these notions as enriched algebraic theories,
- the encoding of the notions into equational systems, and
- properties of theirs obtainable from the theory of equational systems.

5.1 Algebras for pi-calculus

We briefly discuss the concept of π-algebras, an algebraic model of the finitary π-calculus introduced by Stark in [Stark 2005], as algebras for an equational system. The existence of free models is deduced from the theory of equational systems.

We need consider the presheaf category $\mathbf{Set}^{\mathbb{I}}$, for $\mathbb{I}$ the (essentially small) category of finite sets and injections. The category carries an affine doubly closed structure (see [Pym 2002]): the cartesian closed structure $\big(1, \times, (=)^{(-)}\big)$ and the symmetric monoidal closed structure $\big(1, \otimes, (-) \multimap (=)\big)$ with the unit 1 being terminal. The symmetric monoidal closed structure is induced from the symmetric monoidal structure $(\emptyset, \uplus)$ on $\mathbb{I}$ by Day's construction [Day 1970] for $\uplus$ the disjoint-union tensor on $\mathbb{I}$. As the unit for the tensor $\otimes$ is a terminal object, it has the projection maps

$$\begin{aligned} \mathsf{p}_1 &: \quad X \otimes Y \xrightarrow{X \otimes !} X \otimes 1 \xrightarrow{\cong} X \\ \mathsf{p}_2 &: \quad X \otimes Y \xrightarrow{! \otimes Y} 1 \otimes Y \xrightarrow{\cong} Y \end{aligned}$$

and the natural transformation $\mathit{onto}_{X,Y} : Y^X \stackrel{.}{\to} (X \multimap Y)$ given as the transpose of the composite

$$Y^X \otimes X \xrightarrow{\langle \mathsf{p}_1, \mathsf{p}_2 \rangle} Y^X \times X \xrightarrow{\epsilon} Y \ .$$

The *presheaf of names* $N \in \mathbf{Set}^{\mathbb{I}}$ is the inclusion of $\mathbb{I}$ into $\mathbf{Set}$ (*i.e.*, the functor sending a finite set $s \in \mathbb{I}$ to the same set $s \in \mathbf{Set}$).

A π-*algebra* is given by an object $A \in \mathbf{Set}^{\mathbb{I}}$ together with operations $\mathsf{nil} : 1 \to A$, $\mathsf{choice} : A^2 \to A$, $\mathsf{out} : N \times N \times A \to A$, $\mathsf{in} : N \times A^N \to A$, $\mathsf{tau} : A \to A$, and $\mathsf{new} : (N \multimap A) \to A$ satisfying the equations of [Stark 2008, Sections 3.1–3.3 and 3.5]. These algebras, and their homomorphisms, form the category $\mathcal{PI}(\mathbf{Set}^{\mathbb{I}})$.

As mentioned in [Stark 2005, 2008], it is difficult to express π-algebras as algebras for an enriched algebraic theory: π-algebras need two enrichments, while enriched algebraic theories treat only one. More specifically, the operation $\mathsf{new} : (N \multimap A) \to A$ is an operation in $\mathbf{Set}^{\mathbb{I}}$ enriched over itself with the monoidal closed structure $(\otimes, \multimap)$; whilst the other operations of π-algebras are operations in $\mathbf{Set}^{\mathbb{I}}$ enriched over itself with the cartesian closed structure. Thus one cannot use enriched algebraic theories to show the existence of free π-algebras, *i.e.*, that of a left adjoint to the forgetful functor $U_\pi : \mathcal{PI}(\mathbf{Set}^{\mathbb{I}}) \to \mathbf{Set}^{\mathbb{I}}$ taking a π-algebra to its carrier object.

Remark 5.1.1. Using the technique recently developed by Staton [2009], π-algebras can be represented as algebras for an enriched algebraic theory based on a special category other than $\mathbf{Set}^{\mathbb{I}}$.

On the other hand, π-algebras are directly represented as algebras for an equational system. The operations and the equations for π-algebras yield a functorial signature and functorial equations as follows. The functorial signature Σ_π on $\mathbf{Set}^{\mathbb{I}}$ is given by setting

$$\Sigma_\pi(A) = 1 + A^2 + (N \times N \times A) + (N \times A^N) + A + (N \multimap A)\,.$$

In [Stark 2008], the equations for π-algebras are given as certain commuting diagrams. It is easily seen that those commuting diagrams, by uncurryfication, directly define functorial equations. As an example, we consider the equation establishing the inactivity of a process that inputs on a restricted channel:

$$\begin{tikzcd}
A^N \arrow[r, "!"] \arrow[d, "\widehat{\mathsf{in}}"'] & 1 \arrow[r, "\mathsf{nil}"] & A \arrow[d, equal] \\
A^N \arrow[r, "onto"] & N \multimap A \arrow[r, "\mathsf{new}"] & A
\end{tikzcd}$$

where the map $\widehat{\mathsf{in}} : A^N \to A^N$ denotes the transpose of the operation $\mathsf{in} : N \times A^N \to A$. The commuting diagram directly yields a pair of functors $\Sigma_\pi\text{-}\mathbf{Alg} \rightrightarrows (-)^N\text{-}\mathbf{Alg}$ over $\mathbf{Set}^{\mathbb{I}}$. The functoriality of these functorial terms holds because *onto* is a natural transformation.

The functorial signature Σ_π and the functorial equations induced from the axioms of π-algebras constitute an equational system $\mathbb{S}_\pi$ on $\mathbf{Set}^{\mathbb{I}}$ such that $\mathbb{S}_\pi\text{-}\mathbf{Alg} \cong \mathcal{PI}(\mathbf{Set}^{\mathbb{I}})$. From the fact that the presheaves N and 2 are finitely presentable in $\mathbf{Set}^{\mathbb{I}}$, it follows that the functorial signature and equation of $\mathbb{S}_\pi$ are finitary (or equivalently, κ-cocontinuous for every infinite limit ordinal κ). Since the equational system $\mathbb{S}_\pi$ is κ-finitary for every infinite limit ordinal κ, the following result follows from Theorem 4.2.1.

Proposition 5.1.2. *The category of π-algebras $\mathcal{PI}(\mathbf{Set}^{\mathbb{I}}) \cong \mathbb{S}_\pi\text{-}\mathbf{Alg}$ is cocomplete and monadic over $\mathbf{Set}^{\mathbb{I}}$ with the induced monad being finitary.*

The above discussion also applies more generally to axiomatic settings as in [Fiore et al. 2002] and, in particular, to π-algebras over nominal sets, $\omega\mathbf{Cpo}^{\mathbb{I}}$, *etc.*

5.2 Algebras with monoid structure

Following [Fiore et al. 1999, Fiore 2008], we introduce the concept of Σ-monoid, for an endofunctor Σ with a pointed strength, and consider it from the point of view of equational systems. The theory of equational systems is then used to provide an explicit description of free Σ-monoids. We then show that, for Σ_λ the functorial signature of the lambda calculus, the $\beta\eta$ identities are straightforwardly expressible as functorial equations. The theory of equational systems is further used to relate the arising algebraic models by adjunctions.

5.2.1 Σ-monoids

Let Σ be an endofunctor on a monoidal category $\mathscr{C} = (\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$. A *pointed strength* for Σ is a natural transformation

$$\mathsf{st}_{X,(Y,y:I\to Y)} \;:\; \Sigma(X)\otimes Y \xrightarrow{\cdot} \Sigma(X\otimes Y) \;:\; \mathscr{C}\times(I/\mathscr{C}) \to \mathscr{C}$$

satisfying coherence conditions analogous to those of strengths [Kock 1972]:

$$\begin{tikzcd}
\Sigma(A)\otimes I \arrow[r, "\mathsf{st}_{A,(I,\mathrm{id}_I:I\to I)}"] \arrow[dr, "\rho_{\Sigma(A)}"'] & \Sigma(A\otimes I) \arrow[d, "\Sigma(\rho_A)"] \\
 & \Sigma(A)
\end{tikzcd}$$

$$\begin{tikzcd}
(\Sigma(A)\otimes B)\otimes C \arrow[r, "\mathsf{st}_{A,(B,b:I\to B)}\otimes C"] \arrow[d, "\alpha_{\Sigma(A),B,C}"] & \Sigma(A\otimes B)\otimes C \arrow[r, "\mathsf{st}_{A\otimes B,(C,c:I\to C)}"] & \Sigma((A\otimes B)\otimes C) \arrow[d, "\Sigma(\alpha_{A,B,C})"] \\
\Sigma(A)\otimes(B\otimes C) \arrow[rr, "\mathsf{st}_{A,(B\otimes C,(b\otimes c)\circ\rho_I^{-1}:I\to B\otimes C)}"] & & \Sigma(A\otimes(B\otimes C))
\end{tikzcd}$$

commute for all $A\in\mathscr{C}$ and $(B, b: I\to B), (C, c: I\to C)\in I/\mathscr{C}$.

Remark 5.2.1. The notion of pointed strength arises as a special case of that of strength for actions of monoidal categories (see [Fiore 2008] and also [Janelidze and Kelly 2001] and Section 6.1).

For an endofunctor Σ with a pointed strength st on a monoidal category $\mathscr{C}$, the category of Σ-*monoids* Σ-$\mathbf{Mon}(\mathscr{C})$ has objects given by quadruples (X, s, m, e) where $(X, s:\Sigma X\to X)$ is a Σ-algebra and $(X, m: X\otimes X\to X, e: I\to X)$ is a monoid in $\mathscr{C}$ satisfying a compatibility law, which requires that

$$\begin{tikzcd}
\Sigma(X)\otimes X \arrow[r, "\mathsf{st}_{X,(X,e:I\to X)}"] \arrow[d, "s\otimes X"'] & \Sigma(X\otimes X) \arrow[r, "\Sigma(m)"] & \Sigma(X) \arrow[d, "s"] \\
X\otimes X \arrow[rr, "m"] & & X
\end{tikzcd}$$

should commute; morphisms are maps of $\mathscr{C}$ which are both Σ-algebra and monoid homomorphisms.

5.2.2 Equational system and free construction for Σ-monoids

There are some problems with turning Σ-monoids into algebras for an enriched algebraic theory. First, if the monoidal category $\mathscr{C}$ is not closed, then $\mathscr{C}$ is not enriched over itself. More importantly, even when $\mathscr{C}$ is closed, one cannot express the operation $m : X \otimes X \to X$ as an operation in an enriched algebraic theory in general. However, equational systems are free from these problems.

Let $\mathscr{C} = (\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category with binary coproducts. For an endofunctor Σ on $\mathscr{C}$ with a pointed strength st, the equational system $\mathbb{M}_\Sigma$ of Σ-monoids is defined as

$$(\mathscr{C} : F_\Sigma \rhd G_\Sigma \vdash L_\Sigma \equiv R_\Sigma)$$

with

$$\begin{aligned}
F_\Sigma(X) &= \Sigma(X) + (X \otimes X) + I \\
G_\Sigma(X) &= \big((X \otimes X) \otimes X\big) + (I \otimes X) + (X \otimes I) + (\Sigma(X) \otimes X) \\
&L_\Sigma(X, [s, m, e]) \\
&= (X, [\ m \circ (m \otimes \mathrm{id}_X)\ ,\ \lambda_X\ ,\ \rho_X\ ,\ m \circ (s \otimes \mathrm{id}_X)\ ]) \\
&R_\Sigma(X, [s, m, e]) \\
&= (X, [\ m \circ (\mathrm{id}_X \otimes m) \circ \alpha_{X,X,X}\ ,\ m \circ (e \otimes \mathrm{id}_X)\ ,\ m \circ (\mathrm{id}_X \otimes e)\ ,\ s \circ \Sigma(m) \circ \mathsf{st}_{X,(X,e)}\ ])\,.
\end{aligned}$$

The functoriality of L_Σ and R_Σ follow from the naturality of α, λ, ρ and st. By construction, $\mathbb{M}_\Sigma$-**Alg** is (isomorphic to) Σ-**Mon**$(\mathscr{C})$.

Consequently, one can apply the theory of equational systems developed in the previous chapters to the algebra of Σ-monoids. For instance, if $\mathscr{C}$ is cocomplete, and the endofunctor $\Sigma : \mathscr{C} \to \mathscr{C}$ and the tensor product $\otimes : \mathscr{C}^2 \to \mathscr{C}$ are ω-cocontinuous and epicontinuous, then the equational system $\mathbb{M}_\Sigma$ is ω-inductive. Thus, by Theorem 4.2.1, it follows that Σ-**Mon**$(\mathscr{C})$ is monadic over $\mathscr{C}$ and that free Σ-monoids on objects in $\mathscr{C}$ can be constructed as in the diagram (3.3) followed by (3.4).

While this provides a categorical construction of free Σ-monoids, when the monoidal structure is closed, we can go further and give a more explicit description of free Σ-monoids exploiting the following fact.

- When $\mathscr{C}$ is monoidal closed, if the initial $(I + \Sigma(-))$-algebra $\mu X.\, I + \Sigma X$ exists, then the initial Σ-monoid exists and has carrier object $\mu X.\, I + \Sigma X$ equipped with an appropriate Σ-monoid structure (see [Fiore et al. 1999]).

Assume that the monoidal category $\mathscr{C}$ is (right-)closed. By Proposition 3.3.3, a free Σ-monoid over $A \in \mathscr{C}$ is an initial $\mathbb{M}^A_\Sigma$-algebra for the equational system

$$\mathbb{M}^A_\Sigma = \big(\mathscr{C} : (A + F_\Sigma(-)) \rhd G_\Sigma \vdash L_\Sigma U_A \equiv R_\Sigma U_A\big)\ ,$$

where U_A denotes the forgetful functor $(A + F_\Sigma(-))$-**Alg** $\to F_\Sigma$-**Alg**. Furthermore, for the endofunctor $(A \otimes -) + \Sigma(-)$ on $\mathscr{C}$ with the pointed strength given by the composite

$$\begin{aligned}&\big((A \otimes X) + \Sigma(X)\big) \otimes Y \\ &\cong \ \big((A \otimes X) \otimes Y\big) + \Sigma(X) \otimes Y \xrightarrow{\alpha_{A,X,Y} + \mathsf{st}_{X,(Y,y)}} \big(A \otimes (X \otimes Y)\big) + \Sigma(X \otimes Y)\ ,\end{aligned}$$

one can easily establish the isomorphism $p : \mathbb{M}^A_\Sigma$-**Alg** $\cong \mathbb{M}_{(A\otimes -)+\Sigma(-)}$-**Alg** $: q$ with the maps p and q given by

$$\begin{aligned}&p(X,\ [a,s,m,e] : A + \Sigma X + X \otimes X + I \longrightarrow X) \\ &\quad = \ (X,\ [m \circ (a \otimes \mathrm{id}_X), s, m, e] : A \otimes X + \Sigma X + X \otimes X + I \longrightarrow X) \\ &q(X,\ [b,s,m,e] : A \otimes X + \Sigma X + X \otimes X + I \longrightarrow X) \\ &\quad = \ (X,\ [b \circ (\mathrm{id}_A \otimes e) \circ \rho_A^{-1}, s, m, e] : A + \Sigma X + X \otimes X + I \longrightarrow X)\ .\end{aligned}$$

Thus, we have the following result (see also [Fiore 2008]).

- When the monoidal category $\mathscr{C}$ is closed, for any object $A \in \mathscr{C}$, if the initial $\big(I + (A \otimes -) + \Sigma(-)\big)$-algebra $\mu X.\, I + A \otimes X + \Sigma X$ exists, then the free Σ-monoid on A exists and has carrier object $\mu X.\, I + A \otimes X + \Sigma X$ equipped with an appropriate Σ-monoid structure.

5.2.3 Lambda-calculus algebras

As a concrete example, we consider the λ-calculus, whose models are given as Σ-monoids on the presheaf category $\mathbf{Set}^{\mathbb{F}}$, for $\mathbb{F}$ the (essentially small) category of finite sets and functions.

We quickly review the structure of $\mathbf{Set}^{\mathbb{F}}$. Besides the cartesian closed structure, the presheaf category $\mathbf{Set}^{\mathbb{F}}$ is equipped with the substitution monoidal structure $(\bullet, V)$, where the unit V is the embedding of $\mathbb{F}$ into **Set** given by $V(n) = n$ for each $n \in \mathbb{F}$, and the tensor $\bullet$ is given by the coend formula

$$(X \bullet Y)(n) = \int^{k \in \mathbb{F}} X(k) \times (Yn)^k\ .$$

This substitution monoidal structure is closed. We call the exponentiation $(-)^V$ on $\mathbf{Set}^{\mathbb{F}}$ (*i.e.*, the right adjoint to $(-) \times V$) the *shift functor* because, for any presheaf $X \in \mathbf{Set}^{\mathbb{F}}$ and finite set $n \in \mathbb{F}$, the set $X^V(n)$ can be presented as $X(n+1)$.

A λ-prealgebra [Fiore et al. 1999] is a Σ_λ-monoid for the endofunctor Σ_λ given by

$$\Sigma_\lambda X = X^V + X^2$$

with a suitable pointed strength on the presheaf category $\mathbf{Set}^{\mathbb{F}}$. The components of a Σ_λ-monoid

$$(X,\ [\mathsf{abs}, \mathsf{app}, \mathsf{sub}, \mathsf{var}] : X^V + X^2 + (X \bullet X) + V \longrightarrow X)$$

provide interpretations of λ-abstraction ($\mathsf{abs} : X^V \to X$), application ($\mathsf{app} : X^2 \to X$), capture-avoiding simultaneous substitution ($\mathsf{sub} : X \bullet X \to X$), and variables ($\mathsf{var} : V \to X$). The initial Σ_λ-monoid has carrier object $\mu X.\, V + X^V + X^2$, which consists of α-equivalence classes of λ-terms with variables from V, and thus provides an abstract notion of syntax for the λ-calculus (see [Fiore et al. 1999]). The free Σ_λ-monoid on a presheaf $M \in \mathbf{Set}^{\mathbb{F}}$ has carrier object $\mu X.\, V + (M \bullet X) + X^V + X^2$, and provides an abstract syntax for the λ-calculus with (first-order) variables from V and second-order variables from M. This syntactic description of free Σ_λ-monoids has been considered in [Hamana 2004, Fiore 2008].

The $\beta\eta$ identities for a λ-prealgebra on X are expressed, in the internal language, by the following equations

$$
\begin{array}{ll}
(\beta) & f : X^V, x : X \;\vdash\; \mathsf{app}(\mathsf{abs}(f), x) \;=\; \mathsf{sub}(f\langle x\rangle) \;:\; X \\
(\eta) & x : X \;\vdash\; \mathsf{abs}\big(\lambda v : V.\, \mathsf{app}(x, \mathsf{var}\, v)\big) \;=\; x \;:\; X
\end{array}
$$

where the map $-\langle = \rangle : X^V \times X \to X \bullet X$ embeds $X^V \times X$ into $X \bullet X$. Indeed, the equations stand for the following commuting diagrams:

$$
\begin{tikzcd}
X^V \times X \arrow[r, "-\langle = \rangle"] \arrow[d, "\mathsf{abs} \times X"'] & X \bullet X \arrow[d, "\mathsf{sub}"] \\
X \times X \arrow[r, "\mathsf{app}"] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "x:X \,\vdash\, \lambda v:V.\, \mathsf{app}(x, \mathsf{var}\, v)"] \arrow[dr, "\mathrm{id}_X"'] & X^V \arrow[d, "\mathsf{abs}"] \\
& X
\end{tikzcd}
$$

where the map $x : X \;\vdash\; \lambda v : V.\, \mathsf{app}(x, \mathsf{var}\, v)$ is the transpose of the composite

$$
X \times V \xrightarrow{X \times \mathsf{var}} X \times X \xrightarrow{\mathsf{app}} X \; .
$$

These commuting diagrams provide a functorial equation

$$
L_{\beta\eta} = R_{\beta\eta} : F_{\Sigma_\lambda}\text{-}\mathbf{Alg} \to G_{\beta\eta}\text{-}\mathbf{Alg} \; ,
$$

for $G_{\beta\eta}(X) = (X^V \times X) + X$, and yield the equational system of λ-terms

$$
\mathbb{M}_{\Sigma_\lambda}/_{\beta\eta} = (\mathbf{Set}^{\mathbb{F}} : F_{\Sigma_\lambda} \rhd (G_{\Sigma_\lambda} + G_{\beta\eta}) \vdash [L_{\Sigma_\lambda}, L_{\beta\eta}] \equiv [R_{\Sigma_\lambda}, R_{\beta\eta}])
$$

from $\mathbb{M}_{\Sigma_\lambda} = (\mathbf{Set}^{\mathbb{F}} : F_{\Sigma_\lambda} \rhd G_{\Sigma_\lambda} \vdash L_{\Sigma_\lambda} \equiv R_{\Sigma_\lambda})$.

From the coend formula of the substitution tensor $\bullet$ and the fact that filtered colimits commute with finite limits in $\mathbf{Set}$, it follows that the tensor $\bullet : \mathbf{Set}^{\mathbb{F}} \times \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$ preserves filtered colimits. Also, from the coend formula, the tensor $\bullet$ is easily shown to be epicontinuous. It is not hard to show that the endofunctors $(-)^V$ and $(-)^2$ are epicontinuous and ω-cocontinuous. Hence the endofunctors F_{Σ_λ}, G_{Σ_λ}, and $G_{\beta\eta}$ are epicontinuous and ω-cocontinuous. Thus the equational systems $\mathbb{M}_{\Sigma_\lambda}$ and $\mathbb{M}_{\Sigma_\lambda}/_{\beta\eta}$ are ω-inductive. From one application of Theorem 3.2.6 and two applications of Theorem 3.2.8, we obtain the adjunctions $V \dashv U$, $K_1 \dashv J_1$ and $K_{1,2} \dashv J_1\,J_2$, and thus we have that $K_2 = K_{1,2}\,J_1 \dashv J_2$ as in the diagram below:

$$\begin{tikzcd}
\mathbb{M}_{\Sigma_\lambda}/_{\beta\eta}\text{-}\mathbf{Alg} \arrow[r, hook, bend right, "J_2"'] \arrow[r, phantom, "\perp"] & \mathbb{M}_{\Sigma_\lambda}\text{-}\mathbf{Alg} \arrow[l, bend right, "K_2"'] \arrow[r, hook, bend right, "J_1"'] \arrow[r, phantom, "\perp"] & F_{\Sigma_\lambda}\text{-}\mathbf{Alg} \arrow[l, bend right, "K_1"'] \arrow[ll, bend right=40, "K_{1,2}"'] \arrow[d, shift left, "U"] \arrow[d, phantom, "\dashv"] \\
& & \mathbf{Set}^{\mathbb{F}} \arrow[u, shift left, "V"]
\end{tikzcd}$$

Further, by examining the construction (3.4), given in Section 3.2.3, of the free $\mathbb{M}_{\Sigma_\lambda}/_{\beta\eta}$-algebra over the initial $\mathbb{M}_{\Sigma_\lambda}$-algebra along $K_{1,2}\,J_1$, one sees that the presheaf of α-equivalence classes of λ-terms is first quotiented by the $\beta\eta$ identities, and then by the congruence rules for the operations abs, app, and sub. It follows that the initial $\mathbb{M}_{\Sigma_\lambda}/_{\beta\eta}$-algebra is the presheaf of $\alpha\beta\eta$-equivalence classes of λ-terms.

Part II

Term Equational Systems and Equational Reasoning

Chapter 6

Term equational systems

We introduce a notion of system of equations, called *Term Equational System (TES)*, which is more concrete than that of Equational System (ES) but more general than that of enriched algebraic theory, in the sense that one can represent every TES as an ES, and every enriched algebraic theory as a TES. Thus the theory developed for ESs in Part I, such as the construction of free algebras, also applies to TESs. Moreover, we present equational logics for TESs in Chapter 7, which is the main purpose of the development of TESs.

In Section 6.1, we review the notion of action of a monoidal category (see *e.g.* [Janelidze and Kelly 2001]), to be used through Part II, and study its relationship with the well-known notion of enriched category (see *e.g.* [Kelly 1982]).

In Section 6.2, we motivate the definition and the model theory of term equational system.

In Section 6.3, the notion of term equational system is defined. TESs embody a semantic universe together with notions of abstract syntax and equation. A semantic universe for TESs, called *TES-universe*, is technically a category equipped with a biclosed action of a monoidal category. Main examples of TES-universes are tensored and cotensored categories enriched over monoidal closed categories. More familiar examples are symmetric monoidal closed categories.

A notion of abstract syntax for TESs, called *TES-syntax*, on a TES-universe is given by a monad equipped with a *strength* on the underlying category of the TES-universe. In particular, when a TES-universe is given as an enriched tensored and cotensored category (resp. as a symmetric monoidal closed category), the notion of TES-syntax on it amounts to that of enriched monad (resp. that of strong monad in the sense of [Kock 1972]).

Given a TES-syntax with underlying monad $\mathbf{T} = (T, \eta, \mu)$ on a TES-universe with underlying category $\mathscr{C}$, the object $TA \in \mathscr{C}$ for $A \in \mathscr{C}$ intuitively consists of terms with variables in A. As it is well-established from categorical logic (see *e.g.* [Mac Lane and Moerdijk 1992]), we consider generalized elements of the object TA (*i.e.*, morphisms of the form $C \to TA$ for $C \in \mathscr{C}$) as a concrete notion of term, called *TES-term*, of arity A and coarity C. A pair of TES-terms with the same arity and coarity defines a *TES-equation*. Finally, a *term equational system* is given by a TES-universe, a TES-syntax on it and a set of TES-equations for the TES-syntax.

In Section 6.4, a model theory for TESs is presented; that is, a notion of model for TESs and a satisfaction relation between models and TES-equations are given. In particular, if a TES is given by a symmetric monoidal closed category $(\mathscr{C}, \otimes, [-,=])$, a strong monad $\mathbf{T} = (T, \eta, \mu, \mathsf{st})$ on it and a set of TES-equations E, then models for the TES are Eilenberg-Moore algebras $(X, s : TX \to X)$ for the monad $\mathbf{T}$ *satisfying* the equations in E, in the sense that the following diagram commutes for every TES-equation $(u \equiv v : C \to TA) \in E$:

$$
\begin{tikzcd}
{[A,X] \otimes C} \arrow[r, shift left, "{[A,X]\otimes t_1}"] \arrow[r, shift right, "{[A,X]\otimes t_2}"'] & {[A,X] \otimes \mathbf{T}A} \arrow[r, "\mathsf{st}_{[A,X],A}"] & \mathbf{T}([A,X] \otimes A) \arrow[r, "\mathbf{T}(\epsilon^A_X)"] & \mathbf{T}X \arrow[r, "s"] & X
\end{tikzcd}
$$

6.1 Actions of monoidal categories

We briefly review the notion of action of a monoidal category and study its relationship to the notion of enriched category.

6.1.1 Actions and strengths

The notion of action of a monoidal category is a generalization of the usual concept of monoid action. Indeed, an action of a discrete monoidal category on a discrete category amounts to an action of a monoid on a set.

Let $(\mathscr{V}, \cdot, I, \alpha, \lambda, \rho)$ be a monoidal category. A *$\mathscr{V}$-action* $(\mathscr{C}, *, \widetilde{\alpha}, \widetilde{\lambda})$ is given by a category $\mathscr{C}$, a functor $* : \mathscr{V} \times \mathscr{C} \to \mathscr{C}$ and natural isomorphisms $\widetilde{\lambda}_C : I * C \xrightarrow{\cong} C$ and $\widetilde{\alpha}_{U,V,C} : (U \cdot V) * C \xrightarrow{\cong} U * (V * C)$, subject to the following coherence conditions:

$$
\begin{tikzcd}
(I \cdot V) * C \arrow[r, "\widetilde{\alpha}_{I,V,C}"] \arrow[dr, "\lambda_V * C"'] & I * (V * C) \arrow[d, "\widetilde{\lambda}_{V*C}"] \\
& V * C
\end{tikzcd}
\qquad
\begin{tikzcd}
(V \cdot I) * C \arrow[r, "\widetilde{\alpha}_{V,I,C}"] \arrow[dr, "\rho_V * C"'] & V * (I * C) \arrow[d, "V * \widetilde{\lambda}_C"] \\
& V * C
\end{tikzcd}
$$

$$
\begin{tikzcd}
((U \cdot V) \cdot W) * C \arrow[r, "\alpha_{U,V,W} * C"] \arrow[d, "\widetilde{\alpha}_{U \cdot V, W, C}"] & (U \cdot (V \cdot W)) * C \arrow[r, "\widetilde{\alpha}_{U, V \cdot W, C}"] & U * ((V \cdot W) * C) \arrow[d, "U * \widetilde{\alpha}_{V,W,C}"] \\
(U \cdot V) * (W * C) \arrow[rr, "\widetilde{\alpha}_{U,V,W*C}"] & & U * (V * (W * C))
\end{tikzcd}
$$

A *$\mathscr{V}$-strong functor* (F, st) from a $\mathscr{V}$-action $(\mathscr{C}, *, \widetilde{\alpha}, \widetilde{\lambda})$ to another one $(\mathscr{D}, *', \widetilde{\alpha}', \widetilde{\lambda}')$ consists of a functor $F : \mathscr{C} \to \mathscr{D}$ between the underlying categories and a *$\mathscr{V}$-strength* st for the functor F; *i.e.*, a natural transformation $\mathsf{st}_{V,C} : V *' FC \to F(V * C) : \mathscr{V} \times \mathscr{C} \to \mathscr{D}$, subject to the following coherence conditions:

$$
\begin{tikzcd}
I *' FC \arrow[r, "\mathsf{st}_{I,C}"] \arrow[dr, "\widetilde{\lambda}'_{FC}"'] & F(I * C) \arrow[d, "F(\widetilde{\lambda}_C)"] \\
& FC
\end{tikzcd}
\qquad
\begin{tikzcd}
(U \cdot V) *' FC \arrow[r, "\widetilde{\alpha}'_{U,V,FC}"] \arrow[d, "\mathsf{st}_{U \cdot V,C}"] & U *' (V *' FC) \arrow[r, "U *' \mathsf{st}_{V,C}"] & U *' F(V * C) \arrow[d, "\mathsf{st}_{U,V*C}"] \\
F((U \cdot V) * C) \arrow[rr, "F(\widetilde{\alpha}_{U,V,C})"] & & F(U * (V * C))
\end{tikzcd}
$$

Note that the strength need not be a natural isomorphism, and hence that it is a lax notion of morphism between $\mathscr{V}$-actions.

A *$\mathscr{V}$-strong monad* $(\mathbf{T}, \mathsf{st})$ on a $\mathscr{V}$-action $(\mathscr{C}, *)$ consists of a monad $\mathbf{T} = (T, \eta, \mu)$ on the underlying category $\mathscr{C}$ and a *$\mathscr{V}$-strength* st for the monad $\mathbf{T}$; *i.e.*, a *$\mathscr{V}$-strength* st for the underlying functor T satisfying two further coherence conditions:

$$
\begin{tikzcd}
V * TC \arrow[r, "\mathsf{st}_{V,C}"] & T(V * C) \\
V * C \arrow[u, "V * \eta_C"] \arrow[ur, "\eta_{V*C}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
V * TTC \arrow[r, "\mathsf{st}_{V,TC}"] \arrow[d, "V * \mu_C"] & T(V * TC) \arrow[r, "T(\mathsf{st}_{V,C})"] & TT(V * C) \arrow[d, "\mu_{V*C}"] \\
V * TC \arrow[rr, "\mathsf{st}_{V,C}"] & & T(V * C)
\end{tikzcd}
$$

A *$\mathscr{V}$-strong functor morphism* $\kappa : (F, \mathsf{st}) \to (G, \mathsf{st}')$ between $\mathscr{V}$-strong functors from $(\mathscr{C}, *)$ to $(\mathscr{D}, *')$ is a natural transformation $\kappa : F \stackrel{\cdot}{\to} G$ between the underlying functors such that the following diagram commutes:

$$
\begin{tikzcd}
V *' FC \arrow[r, "\mathsf{st}_{V,C}"] \arrow[d, "V *' \kappa_C"'] & F(V * C) \arrow[d, "\kappa_{V*C}"] \\
V *' GC \arrow[r, "\mathsf{st}'_{V,C}"] & G(V * C)
\end{tikzcd}
$$

A $\mathscr{V}$-action $(\mathscr{C}, *)$ is called *right-closed*, or just *closed*, if the functors $(-) * C : \mathscr{V} \to \mathscr{C}$ for all $C \in \mathscr{C}$ have right adjoints; these we call *right-homs* and denote by $\underline{\mathscr{C}}(C, -) : \mathscr{C} \to \mathscr{V}$. On the other hand, it is called *left-closed* if the functors $V * (-) : \mathscr{C} \to \mathscr{C}$ for all $V \in \mathscr{V}$ have right adjoints; these we call *left-homs* and denote by $[\![V, -]\!] : \mathscr{C} \to \mathscr{C}$. When a $\mathscr{V}$-action is both left and right closed, it is said to be *bi-closed*.

6.1.2 Relationship to enriched categories

For a monoidal category $\mathscr{V}$, every right-closed $\mathscr{V}$-action $(\mathscr{C}, *)$ induces a $\mathscr{V}$-enriched category, whose hom-objects are given by right-homs $\underline{\mathscr{C}}(-, =)$. The detailed construction is given in Appendix A. Furthermore, one can show the following.

- To give a $\mathscr{V}$-strong functor between right-closed $\mathscr{V}$-actions is equivalent to giving a $\mathscr{V}$-enriched functor between the associated $\mathscr{V}$-enriched categories.
- To give a $\mathscr{V}$-strong monad between right-closed $\mathscr{V}$-actions is equivalent to giving a $\mathscr{V}$-enriched monad between the associated $\mathscr{V}$-enriched categories.

- To give a $\mathscr{V}$-strong functor morphism between $\mathscr{V}$-strong functors from a right-closed $\mathscr{V}$-action to another right-closed one is equivalent to giving a $\mathscr{V}$-enriched natural transformation between the associated $\mathscr{V}$-enriched functors.

We remark that when $\mathscr{V}$ is monoidal closed, the notion of right-closed $\mathscr{V}$-action essentially amounts to that of $\mathscr{V}$-enriched tensored category (see [Janelidze and Kelly 2001, Section 6]). However, in this case, requiring left-closedness for right-closed $\mathscr{V}$-actions is weaker than requiring cotensors for the corresponding $\mathscr{V}$-enriched tensored categories; because the former requires $V * (-)$ to have a right adjoint, but the latter further requires the adjunction to be enriched. The difference between the two conditions vanishes when $\mathscr{V}$ is symmetric monoidal closed. For example, every monoidal bi-closed category $\mathscr{V}$ yields a bi-closed $\mathscr{V}$-action on itself, but not a $\mathscr{V}$-enriched tensored and cotensored category in general unless $\mathscr{V}$ is symmetric.

6.2 Motivation

We motivate the definition and the model theory of term equational system.

Following the spirit of equational systems, we start by considering an endofunctor Σ on a category $\mathscr{C}$ as a signature specifying algebraic operators, and Σ-algebras as interpretations of the operators. Recall that the intuition behind a free Σ-algebra $(\mathbf{T}_\Sigma X, \tau_X)$ on an object X in $\mathscr{C}$, if it exists, is that the carrier object $\mathbf{T}_\Sigma X$ consists of terms built up from operators in the signature Σ and variables in the object X. Thus, requiring the existence of free Σ-algebras, we consider generalized elements of the objects $\mathbf{T}_\Sigma X$ as a concrete notion of terms of arity X. More precisely, a generalized element $t : C \to \mathbf{T}_\Sigma A$ is called a *generalized term of arity A and coarity C*.

Although the above scenario applies in most applications, one may need go beyond it. For instance, the monad representing second-order abstract syntax [Hamana 2004, Fiore 2008] is not a monad induced from free Σ-algebras for an endofunctor Σ, but a monad induced from free Σ-monoids for an endofunctor Σ. Thus, we more generally consider an arbitrary monad $\mathbf{T}$ as a syntax specifying algebraic terms, and Eilenberg-Moore algebras for the monad $\mathbf{T}$ as interpretations of the terms. Note that this generalize the above setting because Σ-algebras for an endofunctor Σ bijectively correspond to Eilenberg-Moore algebras for the monad $\mathbf{T}_\Sigma$ arising from free Σ-algebras.

We now turn to the model theory. One may simply interpret a *generalized equation* (*i.e.*, a pair of generalized terms with the same arity and coarity) as follows. For a monad $\mathbf{T}$ on a category $\mathscr{C}$, an Eilenberg-Moore algebra $(X, s : \mathbf{T}X \to X)$ satisfies an equation $t_1 \equiv t_2 : C \to \mathbf{T}A$ if, for all valuations $v : A \to X$ of A in X, the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, shift left, "t_1"] \arrow[r, shift right, "t_2"'] & \mathbf{T}A \arrow[r, "\mathbf{T}(v)"] & \mathbf{T}X \arrow[r, "s"] & X \end{tikzcd} .$$

Intuitively, the maps t_1 and t_2 point to two terms with variables in A depending parametrically on C, and the terms are evaluated to values in X according to the interpretation v of variables and the interpretation s of operators.

Although the model theory of (multi-sorted) algebraic theories arises in this way, we need a more general notion of model theory for TESs in order to accommodate model theories of other applications such as nominal equational theories. To this end, we first reformulate the above interpretation of equations as follows. For a monad $\mathbf{T}$ on a locally small category $\mathscr{C}$ with small coproducts, an Eilenberg-Moore algebra $(X, s : \mathbf{T}X \to X)$ satisfies an equation $t_1 \equiv t_2 : C \to \mathbf{T}A$ if the following diagram commutes:

$$\coprod_{v\in\mathscr{C}(A,X)} C \underset{\coprod_{v\in\mathscr{C}(A,X)} t_2}{\overset{\coprod_{v\in\mathscr{C}(A,X)} t_1}{\rightrightarrows}} \coprod_{v\in\mathscr{C}(A,X)} \mathbf{T}A \xrightarrow{[\mathbf{T}(\iota_v)]_{v\in\mathscr{C}(A,X)}} \mathbf{T}(\coprod_{v\in\mathscr{C}(A,X)} A) \xrightarrow{\mathbf{T}([v]_{v\in\mathscr{C}(A,X)})} \mathbf{T}X \xrightarrow{s} X$$

where $\mathscr{C}(-,=)$ denotes the homsets of the category $\mathscr{C}$. We observe that the copower $\otimes : \mathbf{Set} \times \mathscr{C} \to \mathscr{C}$, defined by $S \otimes C = \coprod_{s\in S} C$, gives rise to an action of the cartesian category $\mathbf{Set}$ on the category $\mathscr{C}$; that the homsets $\mathscr{C}(-,=)$ forms right-homs of the action with evaluation maps

$$\epsilon_X^A : \mathscr{C}(A,X) \otimes A = \coprod_{v\in\mathscr{C}(A,X)} A \xrightarrow{[v]_{v\in\mathscr{C}(A,X)}} X ;$$

and that the monad $\mathbf{T}$ has the canonical strength

$$\mathsf{st}_{S,A} : S \otimes \mathbf{T}A = \coprod_{s\in S} \mathbf{T}A \xrightarrow{[\mathbf{T}(\iota_s)]_{s\in S}} \mathbf{T}(\coprod_{s\in S} A) = \mathbf{T}(S \otimes A) .$$

From this observation, we see that the above diagram can be turned into the following:

$$\mathscr{C}(A,X) \otimes C \underset{\mathscr{C}(A,X)\otimes t_2}{\overset{\mathscr{C}(A,X)\otimes t_1}{\rightrightarrows}} \mathscr{C}(A,X) \otimes \mathbf{T}A \xrightarrow{\mathsf{st}_{\mathscr{C}(A,X),A}} \mathbf{T}(\mathscr{C}(A,X) \otimes A) \xrightarrow{\mathbf{T}(\epsilon_X^A)} \mathbf{T}X \xrightarrow{s} X$$

Thus, we generalize the model theory by considering this diagram for an arbitrary right-closed action of a monoidal category on the base category $\mathscr{C}$ and an arbitrary strength for the monad $\mathbf{T}$.

In summary, a TES is given by a base category $\mathscr{C}$ equipped with a right-closed action $*$ of a monoidal category $\mathscr{V}$, a monad $\mathbf{T}$ on $\mathscr{C}$ with a strength st, and a set E of generalized equations. We conclude this motivation by emphasizing that the notions of term, equation and algebra only depend on the base category $\mathscr{C}$ and the monad $\mathbf{T}$, but the interpretation of equations depends parametrically on the $\mathscr{V}$-action structure $*$ on $\mathscr{C}$ and the strength st for the monad $\mathbf{T}$.

6.3 Term equational systems

We introduce the notions of TES-universe, TES-syntax and TES-equation that lead to the concept of Term Equational System (TES).

The notion of TES-universe is defined as in the previous motivation with the additional assumption that the action is also left-closed. This will be crucial to our discussion on equational reasoning for TESs to be introduced in Chapter 7.

Definition 6.3.1 (TES-universe)**.** A *TES-universe* $(\mathscr{C}, \mathscr{V}, *)$ consists of a monoidal category $\mathscr{V}$ and a bi-closed $\mathscr{V}$-action $(\mathscr{C}, *)$.

Example 6.3.2. We give examples of TES-universe. Note that the monoidal categories $\mathscr{V}$ of TES-universes need not be closed, though they are so in these examples.

1. Every category $\mathscr{C}$ with small coproducts and products gives rise to the TES-universe $(\mathscr{C}, \mathbf{Set}, *)$ for **Set** with the cartesian structure, where the actions $V * C$, right-homs $\underline{\mathscr{C}}(C, D)$ and left-homs $[\![V, C]\!]$ are respectively given by the coproducts $\coprod_{v \in V} C$, the hom-sets $\mathscr{C}(C, D)$ and the products $\prod_{v \in V} C$.

2. Every monoidal bi-closed category $(\mathscr{C}, \otimes, I, [-, =]_{\mathsf{R}}, [-, =]_{\mathsf{L}})$ induces the TES-universe $(\mathscr{C}, \mathscr{C}, \otimes)$ with right-homs $\underline{\mathscr{C}}(C, D)$ and left-homs $[\![C, D]\!]$ respectively given by $[C, D]_{\mathsf{R}}$ and $[C, D]_{\mathsf{L}}$. In particular, every symmetric monoidal closed category induces a TES-universe, as it can be seen as a monoidal bi-closed category.

3. For $\mathscr{V}$ monoidal closed, every $\mathscr{V}$-enriched category $\mathscr{C}$ with tensor $\otimes$ and cotensor $\pitchfork$ gives rise to the TES-universe $(\mathscr{C}_0, \mathscr{V}, \otimes_0)$ for $\mathscr{C}_0$ and $\otimes_0$ respectively the underlying ordinary category and functor of $\mathscr{C}$ and $\otimes$, where the right-homs $\underline{\mathscr{C}_0}(C, D)$ and left-homs $[\![V, C]\!]$ are respectively given by the hom-objects $\mathscr{C}(C, D)$ and the cotensors $V \pitchfork C$.

4. From a family of TES-universes $\{\, \mathcal{U}_i = (\mathscr{C}_i, \mathscr{V}, *_i) \,\}_{i \in I}$ for a small set I, when $\mathscr{V}$ has I-indexed products, we obtain the product TES-universe $\prod_{i \in I} \mathcal{U}_i = (\mathscr{C}, \mathscr{V}, *)$, where the underlying category $\mathscr{C}$ is given by the product category $\prod_{i \in I} \mathscr{C}_i$ and where the actions $V * \{C_i\}_{i \in I}$, right-homs $\underline{\mathscr{C}}(\{C_i\}_{i \in I}, \{D_i\}_{i \in I})$ and left-homs $[\![V, \{C_i\}_{i \in I}]\!]$ are respectively given by $\{V *_i C_i\}_{i \in I}$, $\prod_{i \in I} \underline{\mathscr{C}_i}(C_i, D_i)$ and $\{[\![V, C_i]\!]_i\}_{i \in I}$.

The last construction is particularly useful for specifying multi-sorted TESs. When $\mathcal{U}$ is a TES-universe for a single-sorted system, the product universe $\prod_{s \in \mathbf{S}} \mathcal{U}$ for a set of sorts $\mathbf{S}$ typically serves as a TES-universe for the $\mathbf{S}$-sorted version of the system (see Section 8.1).

Definition 6.3.3 (TES-syntax)**.** A *TES-syntax* $(\mathbf{T}, \mathsf{st})$ on a TES-universe $(\mathscr{C}, \mathscr{V}, *)$ is given by a $\mathscr{V}$-strong monad $(\mathbf{T}, \mathsf{st})$ on the $\mathscr{V}$-action $(\mathscr{C}, *)$.

Strong monads for TESs commonly arise from free algebras for strong endofunctors as in the proposition below. The proof of the proposition is given at the end of the section.

Proposition 6.3.4. *For a TES-universe $(\mathscr{C}, \mathscr{V}, *)$ and a $\mathscr{V}$-strong endofunctor (F, st) on $\mathscr{C}$, we assume that the forgetful functor $U_F : F\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint and let $\mathbf{T} = (T, \eta, \mu)$ be the induced monad on $\mathscr{C}$. Then, $\mathbf{T}$ becomes a strong monad, with the components of the strength $\widehat{\mathsf{st}}$ given by the unique maps such that*

$$\begin{tikzcd}
V * FTC \arrow[r, "\mathsf{st}_{V,TC}"] \arrow[d, "V*\tau_C"] & F(V * TC) \arrow[r, "F(\widehat{\mathsf{st}}_{V,C})"] & FT(V * C) \arrow[d, "\tau_{V*C}"] \\
V * TC \arrow[rr, dashed, "\exists!\, \widehat{\mathsf{st}}_{V,C}"] & & T(V * C) \\
V * C \arrow[u, "V*\eta_C"] \arrow[urr, "\eta_{V*C}"'] & &
\end{tikzcd} \tag{6.1}$$

commutes, where $(TX, \tau_X : FTX \to TX)$ denotes the free F-algebra on X.

The intuition here is that the object FX consists of operators applied to variables in X (*i.e.*, terms of depth 1 with variables in X), and the strength $\mathsf{st}_{V,X} : V * FX \to F(V * X)$ puts values from V inside the operators in the terms in FX. The object TX consists of arbitrary terms (of any finite depth) inductively constructed from the variables in X and the operators in F. The strength $\widehat{\mathsf{st}} : V * TX \to T(V * X)$ recursively puts values from V inside the operators in the terms in TX according to the strength st.

For a TES-syntax $\mathbf{T} = (T, \eta, \mu, \mathsf{st})$ on a TES-universe $\mathcal{U}$, we regard generalized elements $C \to TA$ of objects TA (*i.e.*, Kleisli maps) as a concrete notion of term with variables in A parameterized by C.

Definition 6.3.5 (TES-term and TES-equation)**.** Let $(\mathscr{C}, \mathscr{V}, *)$ be a TES-universe and $\mathbf{T} = (T, \eta, \mu, \mathsf{st})$ be a TES-syntax on it. A *TES-term of arity A and coarity C* for the TES-syntax $\mathbf{T}$ is a morphism $C \to TA$ in $\mathscr{C}$. A pair of TES-terms $t \equiv t' : C \to TA$ is called a *TES-equation*.

Definition 6.3.6 (Term equational system)**.** A *TES* $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ consists of a TES-universe $(\mathscr{C}, \mathscr{V}, *)$, a TES-syntax $\mathbf{T} = (T, \eta, \mu, \mathsf{st})$ and a set of TES-equations E.

For simplicity, when a TES-syntax is denoted by $\mathbf{T}$, we implicitly assume that the underlying structure of $\mathbf{T}$ is denoted by $(T, \eta, \mu, \mathsf{st})$.

Example 6.3.7 (TESs for algebraic theories)**.** We encode an algebraic theory $\mathbb{T} = (\Sigma, E)$ into a TES. Recall that the signature Σ induces the endofunctor F_Σ on $\mathbf{Set}$, defined by setting $F_\Sigma(X) = \coprod_{\mathsf{o} \in \Sigma} X^{|\mathsf{o}|}$, such that $\Sigma\text{-}\mathbf{Alg} \cong F_\Sigma\text{-}\mathbf{Alg}$. Also, the forgetful functor $F_\Sigma\text{-}\mathbf{Alg} \to \mathbf{Set}$ is monadic and the induced monad $\mathbf{T}_\Sigma = (T_\Sigma, \eta^\Sigma, \mu^\Sigma)$ is given syntactically: for a set V of variables, the set $T_\Sigma(V)$ consists of terms built up from the variables in V and the operators in Σ.

One can induce a bi-closed action of a monoidal category on the category $\mathbf{Set}$ by means of Example 6.3.2 (1) or (2) because $\mathbf{Set}$ has small products and coproducts, and also **Set** is cartesian closed. Indeed, both ways induce the same action of the cartesian category **Set** on the base category **Set**: the cartesian product $\times : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$.

The functor F_Σ has the canonical **Set**-strength $\mathsf{st} : U \times F_\Sigma(V) \to F_\Sigma(U \times V)$ mapping a pair $(u, \iota_{\mathsf{o}}(v_1, \ldots, v_{|\mathsf{o}|}))$ to $\iota_{\mathsf{o}}((u, v_1), \ldots, (u, v_{|\mathsf{o}|}))$. Following the parameterized induction scheme (6.1), the strength $\widehat{\mathsf{st}} : U \times T_\Sigma(V) \to T_\Sigma(U \times V)$ for the monad $\mathbf{T}_\Sigma$ maps a pair (u, t) to the term $t\{v \mapsto (u, v)\}_{v \in V}$ obtained by simultaneously substituting (u, v) for each variable $v \in V$ in the term t.

By definition, each equation $V \vdash l \equiv r$ in E is given by a pair of terms $l, r \in T_\Sigma(V)$, or equivalently, by a pair of maps $\langle\!\langle l \rangle\!\rangle, \langle\!\langle r \rangle\!\rangle : 1 \to T_\Sigma(V)$. Thus, we can encode the algebraic theory $\mathbb{T}$ as the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle = (\mathbf{Set}, \mathbf{Set}, \times, \mathbf{T}_\Sigma, \langle\!\langle E \rangle\!\rangle)$ with the set of TES-equations $\langle\!\langle E \rangle\!\rangle$ given by $\{\, \langle\!\langle l \rangle\!\rangle \equiv \langle\!\langle r \rangle\!\rangle : 1 \to T_\Sigma V \mid (V \vdash l \equiv r) \in E \,\}$.

Proof of Proposition 6.3.4. We prove a more general proposition, *viz.* Proposition 6.3.9, from which Proposition 6.3.4 follows as a corollary. Proposition 6.3.9 is later used in Section 7.2.

Definition 6.3.8. For a monoidal category $\mathscr{V}$ and a strong endofunctor (F, st) on a left-closed $\mathscr{V}$-action $(\mathscr{C}, *)$, we have the endofunctor $[\![V, -]\!]$ on F-**Alg** for each $V \in \mathscr{V}$ defined by setting

$$[\![V, (X, s : FX \to X)]\!] \;=\; ([\![V, X]\!], s' : F[\![V, X]\!] \to [\![V, X]\!])$$

for s' the transpose of $V * F[\![V, X]\!] \xrightarrow{\mathsf{st}_{V,[\![V,X]\!]}} F(V * [\![V, X]\!]) \xrightarrow{F(\epsilon^V_X)} FX \xrightarrow{\ s\ } X$.

Proposition 6.3.9. *For a monoidal category $\mathscr{V}$, let (F, st) be a strong endofunctor on a left-closed $\mathscr{V}$-action $(\mathscr{C}, *)$ and let $\mathscr{A}$ be a full subcategory of F-**Alg** closed under the operation $[\![V, -]\!]$ for every $V \in \mathscr{V}$. We also assume that the forgetful functor $\mathscr{A} \to \mathscr{C}$ has a left adjoint, sending $X \in \mathscr{C}$ to $(TX, \tau_X : FTX \to TX) \in \mathscr{A}$, and let $\mathbf{T} = (T, \eta, \mu)$ be the associated monad.*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, hook] \arrow[dr, ""{name=A}] & F\text{-}\mathbf{Alg} \arrow[d, "U_F"] \\
& \mathscr{C} \arrow[ul, bend left, ""{name=B}] \arrow[from=A, to=B, phantom, "\perp"]
\end{tikzcd}$$

Then, the following hold.

1. *For every $(Y, t : FY \to Y) \in \mathscr{A}$ and map $f : V * X \to Y$ with $V \in \mathscr{V}, C \in \mathscr{C}$, there uniquely exists a map $f^\# : V * TX \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd}
V * F(TX) \arrow[r, "\mathsf{st}_{V,TX}"] \arrow[d, "V * \tau_X"] & F(V * TX) \arrow[r, "F(f^\#)"] & F(Y) \arrow[d, "t"] \\
V * TX \arrow[rr, "\exists!\, f^\#"] & & Y \\
V * X \arrow[u, "V * \eta_X"] \arrow[urr, "f"'] & &
\end{tikzcd} \tag{6.2}$$

2. *$\mathbf{T}$ becomes a strong monad, with the components of the strength $\widehat{\mathsf{st}}$ given by the unique maps such that the following diagram commutes:*

$$
\begin{tikzcd}
V * FTC \arrow[r, "\mathsf{st}_{V,TC}"] \arrow[d, "V*\tau_C"] & F(V * TC) \arrow[r, "F(\widehat{\mathsf{st}}_{V,C})"] & FT(V * C) \arrow[d, "\tau_{V*C}"] \\
V * TC \arrow[rr, dashed, "\exists!\,\widehat{\mathsf{st}}_{V,C}"] & & T(V * C) \\
V * C \arrow[u, "V*\eta_C"] \arrow[urr, "\eta_{V*C}"'] & &
\end{tikzcd}
\tag{6.3}
$$

Proof. **Proof of 1.** For every $(Y, t : FY \to Y) \in \mathscr{A}$ and map $f : V * X \to Y$, the F-algebra $\llbracket V, (Y,t) \rrbracket$ is in $\mathscr{A}$ since the category $\mathscr{A}$ is closed under the operation $\llbracket V, - \rrbracket$. Thus, by the universal property of the adjunction, there uniquely exists a map $f^{\#} : V * TX \to Y$ making the following diagram commutative:

$$
\begin{tikzcd}
F(TX) \arrow[r, "F(\overline{f^{\#}})"] \arrow[d, "\tau_X"'] & F(\llbracket V, Y \rrbracket) \arrow[d, "\llbracket V,(Y,t) \rrbracket^{\diamond}"] \\
TX \arrow[r, "\overline{f^{\#}}"] & \llbracket V, Y \rrbracket \\
X \arrow[u, "\eta_X"] \arrow[ur, "\overline{f}"'] &
\end{tikzcd}
$$

where $\overline{f}$ and $\overline{f^{\#}}$ respectively denote the transposes of the maps f and $f^{\#}$. Recall from Notation 2.2.2 that $A^{\diamond}$ denotes the structure map of an F-algebra A for an endofunctor F.

By transposing the above diagram, we obtain the diagram (6.2) because the following diagram commutes:

$$
\begin{tikzcd}
V * F(TX) \arrow[r, "\mathsf{st}_{V,TX}"] \arrow[d, "V*F(\overline{f^{\#}})"] & F(V * TX) \arrow[r, "F(f^{\#})"] \arrow[d, "F(V*\overline{f^{\#}})"'] & F(Y) \arrow[dd, "t"] \\
V * F(\llbracket V, Y \rrbracket) \arrow[r, "\mathsf{st}_{V,\llbracket V,Y \rrbracket}"] \arrow[d, "V*\llbracket V,(Y,t) \rrbracket^{\diamond}"] & F(V * \llbracket V, Y \rrbracket) \arrow[ur, "F(\epsilon^V_Y)"'] \arrow[d, phantom, "(\mathsf{A})"] & \\
V * \llbracket V, Y \rrbracket \arrow[rr, "\epsilon^V_Y"] & & Y
\end{tikzcd}
$$

where the commutativity of the diagram (A) follows from the definition of the F-algebra $\llbracket V, (Y,t) \rrbracket$. Hence, there exists a unique map $f^{\#} : V * TX \to Y$ such that the diagram (6.2) commutes.

Proof of 2. First, by item 1, we know that such maps $\widehat{\mathsf{st}}_{V,C}$ uniquely exist. Now we need to show that $\widehat{\mathsf{st}}_{V,C}$ is natural in V and C, and satisfies the four coherence conditions of strength.

The naturality of $\widehat{\mathsf{st}}$, *i.e.*, the equality

$$T(f * g) \circ \widehat{st}_{V,C} = \widehat{st}_{V',C'} \circ (f * T(g))$$

for every $f : V \to V'$ in $\mathscr{V}$ and $g : C \to C'$ in $\mathscr{C}$, follows from item 1 and the commutativity of the following two diagrams:

$$
\begin{tikzcd}
V * FTC \arrow[r, "\mathsf{st}_{V,TC}"] \arrow[d, "V*\tau_C"'] & F(V*TC) \arrow[r, "F(\widehat{\mathsf{st}}_{V,C})"] & FT(V*C) \arrow[r, "FT(f*g)"] \arrow[d, "\tau_{V*C}"] & FT(V'*C') \arrow[d, "\tau_{V'*C'}"] \\
V * TC \arrow[rr, dashed, "\widehat{\mathsf{st}}_{V,C}"] & & T(V*C) \arrow[r, "T(f*g)"] & T(V'*C') \\
V * C \arrow[u, "V*\eta_C"] \arrow[urr, "\eta_{V*C}"'] \arrow[rr, "f*g"'] & & V'*C' \arrow[ur, "\eta_{V'*C'}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
V * FTC \arrow[r, "\mathsf{st}_{V,TC}"] \arrow[dr, "f*FT(g)"'] \arrow[dd, "V*\tau_C"'] & F(V*TC) \arrow[r, "F(f*T(g))"] & F(V'*TC') \arrow[r, "F(\widehat{\mathsf{st}}_{V',C'})"] & FT(V'*C') \arrow[dd, "\tau_{V'*C'}"] \\
& V'*FTC' \arrow[ur, "\mathsf{st}_{V',TC'}"'] \arrow[d, "V'*\tau_{C'}"] & & \\
V * TC \arrow[r, "f*T(g)"] & V'*TC' \arrow[rr, "\widehat{\mathsf{st}}_{V',C'}"] & & T(V'*C') \\
V * C \arrow[u, "V*\eta_C"] \arrow[r, "f*g"'] & V'*C' \arrow[u, "V'*\eta_{C'}"] \arrow[urr, "\eta_{V'*C'}"'] & &
\end{tikzcd}
$$

The first coherence equality

$$
T(\widetilde{\lambda}_C) \circ \widehat{\mathsf{st}}_{I,C} = \widetilde{\lambda}_{TC}
$$

follows from item 1 and the commutativity of the following two diagrams:

$$
\begin{tikzcd}
I * FTC \arrow[r, "\mathsf{st}_{I,TC}"] \arrow[d, "I*\tau_C"] & F(I*TC) \arrow[r, "F(\widehat{\mathsf{st}}_{I,C})"] & FT(I*C) \arrow[r, "FT(\widetilde{\lambda}_C)"] \arrow[d, "\tau_{I*C}"] & FTC \arrow[d, "\tau_C"] \\
I * TC \arrow[rr, dashed, "\widehat{\mathsf{st}}_{I,C}"] & & T(I*C) \arrow[r, "T(\widetilde{\lambda}_C)"] & TC \\
I * C \arrow[u, "I*\eta_C"] \arrow[urr, "\eta_{I*C}"'] \arrow[rr, "\widetilde{\lambda}_C"'] & & C \arrow[ur, "\eta_C"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& F(I*TC) \arrow[dr, "F(\widetilde{\lambda}_{TC})"] \arrow[d, phantom, "\text{(A)}"] & \\
I * FTC \arrow[ur, "\mathsf{st}_{I,TC}"] \arrow[rr, "\widetilde{\lambda}_{FTC}"] \arrow[d, "I*\tau_C"] & {} & FTC \arrow[d, "\tau_C"] \\
I * TC \arrow[rr, "\widetilde{\lambda}_{TC}"] & & TC \\
I * C \arrow[u, "I*\eta_C"] \arrow[r, "\widetilde{\lambda}_C"'] & C \arrow[ur, "\eta_C"'] &
\end{tikzcd}
$$

where the diagram (A) commutes by the first coherence condition of the strength st.

The second coherence equality

$$T(\widetilde{\alpha}_{U,V,C}) \circ \widehat{\mathsf{st}}_{U\cdot V,C} = \widehat{\mathsf{st}}_{U,V*C} \circ (U * \widehat{\mathsf{st}}_{V,C}) \circ \widetilde{\alpha}_{U,V,TC}$$

follows from item 1 and the commutativity of the following two diagrams:

$$\begin{tikzcd}
(U\cdot V)*FTC \arrow[r, "\mathsf{st}_{U\cdot V,TC}"] \arrow[d, "(U\cdot V)*\tau_C"] & F((U\cdot V)*TC) \arrow[r, "F(\widehat{\mathsf{st}}_{U\cdot V,C})"] & FT((U\cdot V)*C) \arrow[r, "FT(\widetilde{\alpha}_{U,V,C})"] \arrow[d, "\tau_{(U\cdot V)*C}"] & FT(U*(V*C)) \arrow[d, "\tau_{U*(V*C)}"] \\
(U\cdot V)*TC \arrow[rr, "\widehat{\mathsf{st}}_{U\cdot V,C}"] & & T((U\cdot V)*C) \arrow[r, "T(\widetilde{\alpha}_{U,V,C})"] & T(U*(V*C)) \\
(U\cdot V)*C \arrow[u, "(U\cdot V)*\eta_C"] \arrow[urr, "\eta_{(U\cdot V)*C}"] \arrow[rr, "\widetilde{\alpha}_{U,V,C}"'] & & U*(V*C) \arrow[ur, "\eta_{U*(V*C)}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(U{\cdot}V)*FTC \arrow[r, "\mathsf{st}_{U\cdot V,TC}"] \arrow[dd, "(U\cdot V)*\tau_C"] \arrow[dr, "\widetilde{\alpha}_{U,V,FTC}"] & F((U{\cdot}V)*TC) \arrow[r, "F(\widetilde{\alpha}_{U,V,TC})"] \arrow[d, phantom, "(\mathsf{A})"] & F(U*(V*TC)) \arrow[r, "F(U*\widehat{\mathsf{st}}_{V,C})"] & F(U*T(V*C)) \arrow[r, "F(\widehat{\mathsf{st}}_{U,V*C})"] & FT(U*(V*C)) \arrow[dd, "\tau_{U*(V*C)}"] \\
& U*(V*FTC) \arrow[r, "U*\mathsf{st}_{V,TC}"] \arrow[d, "U*(V*\tau_C)"] & U*F(V*TC) \arrow[u, "\mathsf{st}_{U,V*TC}"] \arrow[r, "U*F(\widehat{\mathsf{st}}_{V,C})"] & U*FT(V*C) \arrow[u, "\mathsf{st}_{U,T(V*C)}"] \arrow[d, "U*\tau_{V*C}"] & \\
(U{\cdot}V)*TC \arrow[r, "\widetilde{\alpha}_{U,V,TC}"] & U*(V*TC) \arrow[rr, "U*\widehat{\mathsf{st}}_{V,C}"] & & U*T(V*C) \arrow[r, "\widehat{\mathsf{st}}_{U,V*C}"] & T(U*(V*C)) \\
(U{\cdot}V)*C \arrow[u, "(U\cdot V)*\eta_C"] \arrow[r, "\widetilde{\alpha}_{U,V,C}"'] & U*(V*C) \arrow[u, "U*(V*\eta_C)"] \arrow[urr, "U*\eta_{V*C}"] \arrow[urrr, "\eta_{U*(V*C)}"'] & & &
\end{tikzcd}$$

where the diagram (A) commutes by the second coherence condition of the strength st.

The third coherence equality

$$\widehat{\mathsf{st}}_{V,C} \circ (V * \eta_C) = \eta_{V*C}$$

is the bottom of the diagram (6.3).

The last coherence equality

$$\widehat{\mathsf{st}}_{V,C} \circ (V * \mu_C) = \mu_{V*C} \circ T(\widehat{\mathsf{st}}_{V,C}) \circ \widehat{\mathsf{st}}_{V,TC}$$

follows from item 1 and the commutativity of the following two diagrams:

$$\begin{tikzcd}
V * FTTC \arrow[r, "\mathsf{st}_{V,TTC}"] \arrow[dd, "V*\tau_{TC}"'] \arrow[dr, "V*F(\mu_C)"'] & F(V * TTC) \arrow[r, "F(V*\mu_C)"] & F(V * TC) \arrow[r, "F(\widehat{\mathsf{st}}_{V,C})"] & FT(V * C) \arrow[dd, "\tau_{V*C}"] \\
\arrow[r, phantom, "(\mathsf{A})"] & V * FTC \arrow[ur, "\mathsf{st}_{V,TC}"'] \arrow[d, "V*\tau_C"] & & \\
V * TTC \arrow[r, "V*\mu_C"] & V * TC \arrow[rr, "\widehat{\mathsf{st}}_{V,C}"] & & T(V * C) \\
V * TC \arrow[u, "V*\eta_{TC}"] \arrow[ur, "\mathrm{id}"] \arrow[u, phantom, shift left=6, "(\mathsf{B})"] \arrow[urrr, "\widehat{\mathsf{st}}_{V,C}"'] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
V * FTTC \arrow[r, "\mathsf{st}_{V,TTC}"] \arrow[d, "V*\tau_{TC}"] & F(V * TTC) \arrow[r, "F(\widehat{\mathsf{st}}_{V,TC})"] & FT(V * TC) \arrow[r, "FT(\widehat{\mathsf{st}}_{V,C})"] \arrow[d, "\tau_{V*TC}"] & FTT(V * C) \arrow[r, "F(\mu_{V*C})"] \arrow[d, "\tau_{T(V*C)}"] \arrow[dr, phantom, "\mathsf{(C)}"] & FT(V * C) \arrow[d, "\tau_{V*C}"] \\
V * TTC \arrow[rr, "\widehat{\mathsf{st}}_{V,TC}"] & & T(V * TC) \arrow[r, "T(\widehat{\mathsf{st}}_{V,C})"] & TT(V * C) \arrow[r, "\mu_{V*C}"] \arrow[dr, phantom, "\mathsf{(D)}"] & T(V * C) \\
V * TC \arrow[u, "V*\eta_{TC}"] \arrow[urr, "\eta_{V*TC}"] \arrow[rrr, "\widehat{\mathsf{st}}_{V,C}"'] & & & T(V * C) \arrow[u, "\eta_{T(V*C)}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

where the commutativity of the diagrams $\mathsf{(A)}$, $\mathsf{(B)}$, $\mathsf{(C)}$ and $\mathsf{(D)}$ follows from the definition of the multiplication $\mu_X : TTX \to TX$ as the unique map satisfying $\mu_X \circ \tau_{TX} = \tau_X \circ F(\mu_X)$ and $\mu_X \circ \eta_{TX} = \mathrm{id}_{TX}$. $\square$

6.4 Model theory

We introduce a model theoretic notion of equality between TES-terms, leading to the notion of models for TESs.

Let $(\mathscr{C}, \mathscr{V}, *)$ be a TES-universe and $\mathbf{T} = (T, \eta, \mu, \mathsf{st})$ a TES-syntax on it. Then, every TES-term $t : C \to TA$ induces a functorial term

$$[\![t]\!] : T\text{-}\mathbf{Alg} \to (\underline{\mathscr{C}}(A, -) * C)\text{-}\mathbf{Alg}$$

over $\mathscr{C}$, mapping $s : TX \to X$ to the composite

$$\underline{\mathscr{C}}(A, X) * C \xrightarrow{\underline{\mathscr{C}}(A,X)*t} \underline{\mathscr{C}}(A, X) * TA \xrightarrow{\mathsf{st}_{\underline{\mathscr{C}}(A,X),A}} T\big(\underline{\mathscr{C}}(A, X) * A\big) \xrightarrow{T(\epsilon^A_X)} TX \xrightarrow{\ s\ } X\ ,$$

where T-**Alg** denotes the category of T-algebras for the underlying endofunctor T. We remark that this map is the transpose of the composite

$$\underline{\mathscr{C}}(A, X) \xrightarrow{\underline{T}_{A,X}} \underline{\mathscr{C}}(TA, TX) \xrightarrow{\underline{\mathscr{C}}(t,s)} \underline{\mathscr{C}}(C, X)$$

for $\underline{T}$ the $\mathscr{V}$-enriched functor corresponding to the $\mathscr{V}$-strong functor (T, st).

The functorial interpretation of TES-terms induces a *satisfaction relation* between T-algebras and TES-equations: for a T-algebra (X, s),

$$(X, s) \models u \equiv v : C \to TA \quad \text{iff} \quad [\![u]\!](X, s)^\diamond = [\![v]\!](X, s)^\diamond : \underline{\mathscr{C}}(A, X) * C \to X\ .$$

Recall from Notation 2.2.2 that $A^\diamond$ denotes the structure map of an F-algebra A for an endofunctor F. More generally, for a set of T-algebras $\mathscr{A}$, we set $\mathscr{A} \models u \equiv v$ iff $(X, s) \models u \equiv v$ for all $(X, s) \in \mathscr{A}$.

Definition 6.4.1 (Algebras for TESs)**.** An $\mathbb{S}$-*algebra* for a TES $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ is an Eilenberg-Moore algebra (X, s) for the monad $\mathbf{T}$ satisfying the equations in E; that is, such that $(X, s) \models u \equiv v$ for all $(u \equiv v) \in E$.

The category $\mathbb{S}$-**Alg** is the full subcategory of $\mathscr{C}^{\mathbf{T}}$ consisting of the $\mathbb{S}$-algebras, for $\mathscr{C}^{\mathbf{T}}$ the category of Eilenberg-Moore algebras for the monad $\mathbf{T}$. We thus have the following situation:

$$\begin{tikzcd}
\mathbb{S}\text{-}\mathbf{Alg} \arrow[r, hook] \arrow[dr, "U_{\mathbb{S}}"'] & \mathscr{C}^{\mathbf{T}} \arrow[r, hook] \arrow[d, "U_{\mathbf{T}}"] & T\text{-}\mathbf{Alg} \arrow[dl, "U_T"] \\
& \mathscr{C} &
\end{tikzcd}$$

Example 6.4.2 (continued)**.** Let $\langle\!\langle \mathbb{T} \rangle\!\rangle = (\mathbf{Set}, \mathbf{Set}, \times, \mathbf{T}_\Sigma, \langle\!\langle E \rangle\!\rangle)$ be the TES associated to an algebraic theory $\mathbb{T} = (\Sigma, E)$. From the model theory of TES, it follows that a $\langle\!\langle \mathbb{T} \rangle\!\rangle$-algebra is an Eilenberg-Moore algebra $(X, s : T_\Sigma X \to X)$ for $\mathbf{T}_\Sigma$ such that the following diagram commutes for every equation $V \vdash t_1 \equiv t_2$ in E:

$$\mathbf{Set}(V,X) \times 1 \underset{\mathbf{Set}(V,X)\times\langle\!\langle t_2\rangle\!\rangle}{\overset{\mathbf{Set}(V,X)\times\langle\!\langle t_1\rangle\!\rangle}{\rightrightarrows}} \mathbf{Set}(V,X) \times T_\Sigma V \xrightarrow{\widehat{\mathrm{st}}_{\mathbf{Set}(V,X),V}} T_\Sigma\big(\mathbf{Set}(V,X) \times V\big) \xrightarrow{T_\Sigma(\epsilon^V_X)} T_\Sigma X \xrightarrow{s} X \,.$$

It can be easily shown that the commutativity of the above diagram amounts to the following:

$$\text{for all functions } v : V \to X, \quad 1 \underset{\langle\!\langle t_2\rangle\!\rangle}{\overset{\langle\!\langle t_1\rangle\!\rangle}{\rightrightarrows}} T_\Sigma V \xrightarrow{T_\Sigma(v)} T_\Sigma X \xrightarrow{\quad s \quad} X \quad \text{commutes.} \tag{6.4}$$

Let $(X, \overline{[\![-]\!]})$ be the Eilenberg-Moore algebra for the monad $\mathbf{T}_\Sigma$ corresponding to a Σ-algebra $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma})$ via the isomorphism $\Sigma\text{-}\mathbf{Alg} \cong \mathscr{C}^{\mathbf{T}_\Sigma}$. Then, it is easily seen that the Eilenberg-Moore algebra $(X, \overline{[\![-]\!]})$ satisfies the above condition (6.4) if and only if the Σ-algebra $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma})$ satisfies the equation $V \vdash t_1 \equiv t_2$. Thus, it follows that $\langle\!\langle \mathbb{T} \rangle\!\rangle$-**Alg** is isomorphic to the category $\mathbb{T}$-**Alg** of algebras for the algebraic theory $\mathbb{T}$.

Note that TES-terms of the more general form $I \to T_\Sigma V$ for an arbitrary set I do not add expressivity to algebraic theories, since they can be equivalently represented as I-indexed families of TES-terms of the form $1 \to T_\Sigma V$. More formally, the following holds for every T_Σ-algebra (X, s):

$$(X,s) \models u \equiv v : I \to T_\Sigma V \qquad \text{iff} \qquad \forall_{i:1\to I} \;\; (X,s) \models u \circ i \equiv v \circ i : 1 \to T_\Sigma V \;.$$

6.5 Representation as equational systems

The model theory of TESs can be easily recast in the framework of equational systems. For $\mathscr{C}$ with small coproducts, every TES $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ induces a monadic equational system

$$\overline{\mathbb{S}} = (\mathscr{C} : \mathbf{T} \rhd \Gamma_E \vdash \overline{L} \equiv \overline{R})$$

such that $\overline{\mathbb{S}}$-**Alg** $\cong$ $\mathbb{S}$-**Alg** (see Section 2.3 for monadic equational system), where

$$\Gamma_E(X) \;=\; \textstyle\coprod_{(u\,\equiv\,v\,:\,C\to TA)\in E} \mathscr{C}(A,X) * C \;,$$

$$\overline{L}(X,s) \;=\; \big[\, [\![u]\!](X,s)^\diamond \,\big]_{(u\,\equiv\,v)\in E} \;, \qquad \overline{R}(X,s) \;=\; \big[\, [\![v]\!](X,s)^\diamond \,\big]_{(u\,\equiv\,v)\in E} \,.$$

Furthermore, when the strong monad $\mathbf{T}$ arises from free algebras for a strong endofunctor F as in Proposition 6.3.4, the TES $\mathbb{S}$ induces a simpler equational system $\widehat{\mathbb{S}}$ with $\widehat{\mathbb{S}}\text{-}\mathbf{Alg} \cong \mathbb{S}\text{-}\mathbf{Alg}$. Indeed, $\widehat{\mathbb{S}}$ is given by $\big(\mathscr{C} : F \rhd \Gamma_E \vdash \widehat{L} \equiv \widehat{R}\big)$, where

$$\widehat{L}(X, s) \;=\; \big[\, [\![u]\!](X, s^*)^\diamond \,\big]_{(u \,\equiv\, v) \in E} \;, \quad \widehat{R}(X, s) \;=\; \big[\, [\![v]\!](X, s^*)^\diamond \,\big]_{(u \,\equiv\, v) \in E}$$

for (X, s^*) the Eilenberg-Moore algebra for $\mathbf{T}$ corresponding to the F-algebra (X, s) via the isomorphism $\mathscr{C}^{\mathbf{T}} \cong F\text{-}\mathbf{Alg}$. This enables us to apply the theory of Part I to construct free algebras for TESs, as we do in the following chapter.

Chapter 7

Equational reasoning for term equational systems

We discuss equational reasoning for TESs, that is to say formal methods for proving the validity of equality judgements for TESs. For this, we consider two different styles of equational reasoning: deductive reasoning (or reasoning by deduction) and rewriting reasoning (or reasoning by rewriting). By a deductive equational reasoning, we mean a system consisting of a set of inference rules of the form

$$\frac{J_1 \ \ldots \ J_n}{J}$$

where the equational judgements $J_1, \ldots, J_n$ are premises and the equational judgement J is a conclusion. In this system, a proof of a judgement is given by a *proof tree*; *i.e.*, a finite tree made up of instances of the inference rules where the goal judgement is placed in the root and empty premises in the leaves.

On the other hand, by a rewriting equational reasoning, we mean a system consisting of a set $\mathcal{R}$ of rewriting rules of the form

$$\Gamma \vdash t \to t'$$

stating that the term t can rewrite to the term t' in the context Γ. In this system, a proof of an equality judgement $\Delta \vdash s \equiv s'$ is given by a sequence of bidirectional rewriting $\Delta \vdash s \leftrightarrow_{\mathcal{R}} s_1$, $\Delta \vdash s_1 \leftrightarrow_{\mathcal{R}} s_2$, $\ldots$, $\Delta \vdash s_n \leftrightarrow_{\mathcal{R}} s'$ for some $n \geq 0$, where $\Delta \vdash u \leftrightarrow_{\mathcal{R}} v$ denotes that either $(\Delta \vdash u \to v)$ or $(\Delta \vdash v \to u)$ is an instance of a rewriting rule in $\mathcal{R}$.

In Section 7.1, a deductive equational reasoning for TESs, called *Term Equational Logic (TEL)*, is introduced. TEL consists of seven inference rules Ref, Sym, Trans, Axiom, Subst, Ext and Local. We show that TEL is sound, in the usual sense that if a TES-equation $u \equiv v$ has a proof in TEL for a TES $\mathbb{S}$, then the equation $u \equiv v$ is *valid*, *i.e.*, satisfied by all $\mathbb{S}$-algebras. We do not have a general completeness result (*i.e.*, the converse of soundness) for TEL.

In the direction of completeness, in Section 7.2 we give an internal completeness result for TESs $\mathbb{S}$ that admit free algebras: a TES-equation $u \equiv v : C \to TA$ is valid if and only if it is satisfied by the free $\mathbb{S}$-algebra on the object A. By the theory of Part I, if the TES satisfies some conditions, we have an explicit categorical construction of the free algebra. Furthermore, the categorical construction together with the internal completeness result—intuitively and informally—can be seen as describing a sound and complete rewriting equational reasoning. For instance, given a concrete TES, one may synthesize a complete rewriting equational reasoning by analyzing the construction of free algebras, as exemplified in Example 7.2.7 and in the applications of Chapter 8.

7.1 Equational reasoning by deduction

We introduce a sound deductive system for reasoning about equality between TES-terms, called *Term Equational Logic (TEL)*, and show its soundness.

7.1.1 Term equational logic

For a TES $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$, we consider equality judgements of the form

$$E \vdash u \equiv v : C \to TA$$

where u, v are TES-terms with arity A and coarity C in $\mathscr{C}$. The deductive system for deriving such judgements, called *Term Equational Logic (TEL)*, consists of the following inference rules.

- Equality rules.

$$\mathsf{Ref}\ \frac{}{E \vdash u \equiv u} \qquad \mathsf{Sym}\ \frac{E \vdash u \equiv v}{E \vdash v \equiv u} \qquad \mathsf{Trans}\ \frac{E \vdash u \equiv v \quad E \vdash v \equiv w}{E \vdash u \equiv w}$$

- Axioms.

$$\mathsf{Axiom}\ \frac{}{E \vdash u \equiv v}\ (u \equiv v) \in E$$

- Congruence of substitution.

$$\mathsf{Subst}\ \frac{E \vdash u_1 \equiv v_1 : C \to TB \qquad E \vdash u_2 \equiv v_2 : B \to TA}{E \vdash u_1\{u_2\} \ \equiv \ v_1\{v_2\} : C \to TA}$$

  where $w_1\{w_2\}$ denotes the Kleisli composite $C \xrightarrow{w_1} TB \xrightarrow{T(w_2)} T(TA) \xrightarrow{\mu_C} TA$.

- Congruence of tensor extension.

$$\mathsf{Ext}\ \frac{E \vdash u \equiv v : C \to TA}{E \vdash \langle V \rangle u \ \equiv \ \langle V \rangle v : V * C \to T(V * A)}\ (V \in \mathscr{V})$$

  where $\langle V \rangle w$ denotes the composite $V * C \xrightarrow{V * w} V * TA \xrightarrow{\mathsf{st}_{V,C}} T(V * A)$.

- Local character.

$$\mathsf{Local}\ \frac{\{\, E \vdash u \circ e_i \;\equiv\; v \circ e_i : C_i \to TA \,\}_{i \in I}}{E \vdash u \equiv v : C \to TA}\ \big(\{\, e_i : C_i \to C \,\}_{i\in I} \text{ jointly epi}\big)$$

Recall that a family of maps $\{\, e_i : C_i \to C \,\}_{i\in I}$ is said to be *jointly epimorphic* if, for any $f, g : C \to X$ such that $\forall_{i\in I}\ f \circ e_i \;=\; g \circ e_i : C_i \to X$, it follows that $f \;=\; g$.

Example 7.1.1 (continued)**.** As algebraic theories arise as TESs, from the term equational logic we can derive an equational logic for algebraic theories. For instance, for an algebraic theory $\mathbb{T} = (\Sigma, E)$, we have the following deduction rules:

$$\mathsf{Ref}\ \frac{}{V \vdash t \equiv t}\ t \in T_\Sigma V \qquad \mathsf{Sym}\ \frac{V \vdash t \equiv t'}{V \vdash t' \equiv t} \qquad \mathsf{Trans}\ \frac{V \vdash t \equiv t' \quad V \vdash t' \equiv t''}{V \vdash t \equiv t''}$$
$$\mathsf{Axiom}\ \frac{}{V \vdash t \equiv t'}\ (V \vdash t \equiv t') \in E \tag{7.1}$$
$$\mathsf{Subst}\ \frac{U \vdash t \equiv t' \qquad \{\, V \vdash s_u \equiv s'_u \,\}_{u\in U}}{V \vdash t\{u \mapsto s_u\}_{u\in U} \;\equiv\; t'\{u \mapsto s'_u\}_{u\in U}}$$

where $t\{u \;\mapsto\; s_u\}_{u\in U}$ denotes the term obtained by simultaneously substituting the terms s_u for the variables $u \in U$ in the term t. The rules Ref, Sym, Trans and Axiom directly follow from the corresponding TES rules. The rule Subst is derived from the TES rules Local and Subst in the following way:

$$\frac{\langle\!\langle t \rangle\!\rangle \equiv \langle\!\langle t' \rangle\!\rangle : 1 \to T_\Sigma U \qquad \dfrac{\{\, \langle\!\langle s_u \rangle\!\rangle \equiv \langle\!\langle s'_u \rangle\!\rangle : 1 \to T_\Sigma V \,\}_{u\in U}}{[\, \langle\!\langle s_u \rangle\!\rangle \,]_{u\in U} \;\equiv\; [\, \langle\!\langle s'_u \rangle\!\rangle \,]_{u\in U} : U \to T_\Sigma V}\ (\text{by } \mathsf{Local})}{\langle\!\langle t \rangle\!\rangle\{\, [\, \langle\!\langle s_u \rangle\!\rangle \,]_{u\in U} \,\} \;\equiv\; \langle\!\langle t' \rangle\!\rangle\{\, [\, \langle\!\langle s'_u \rangle\!\rangle \,]_{u\in U} \,\} : 1 \to T_\Sigma V}\ (\text{by } \mathsf{Subst})$$

Note that the TES rule Ext is redundant here because the interpretation of the rule for algebraic theories basically states that one can have many copies of a valid equality judgement.

The soundness of this logic for algebraic theories follows from the soundness of TEL, to be shown in the next section.

7.1.2 Soundness of term equational logic

The following theorem shows the soundness of term equational logic.

Theorem 7.1.2 (Soundness of TEL)**.** *For a TES* $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$

$$E \vdash u \equiv v \quad \textit{implies} \quad \mathbb{S}\text{-}\mathbf{Alg} \models u \equiv v\,.$$

Proof. We show the soundness of each rule of TEL; that is to say, that every $\mathbb{S}$-algebra satisfying all the premises of a TEL rule also satisfies its conclusion.

The soundness of the rules Ref, Sym, Trans and Axiom trivially holds.

To show the soundness of the rule Subst, we assume that an $\mathbb{S}$-algebra (X, s) satisfies the two equations $u_1 \equiv v_1 : C \to TB$ and $u_2 \equiv v_2 : B \to TA$; that is to say, that

$$\begin{aligned} [\![u_1]\!](X,s)^\diamond = [\![v_1]\!](X,s)^\diamond &\ : \ \underline{\mathscr{C}}(B,X) * C \to X\,, \\ [\![u_2]\!](X,s)^\diamond = [\![v_2]\!](X,s)^\diamond &\ : \ \underline{\mathscr{C}}(A,X) * B \to X\,. \end{aligned}$$

The interpretation map $[\![u_1\{u_2\}]\!](X,s)^\diamond : \underline{\mathscr{C}}(A,X) * C \to X$ of the TES-term $u_1\{u_2\}$ factors as the composite

$$[\![u_1]\!](X,s)^\diamond \circ \big(\overline{[\![u_2]\!](X,s)^\diamond} * C\big)\,,$$

as shown in the commutative diagram below, where $\overline{[\![u_2]\!](X,s)^\diamond} : \underline{\mathscr{C}}(A,X) \to \underline{\mathscr{C}}(B,X)$ is the transpose of the map $[\![u_2]\!](X,s)^\diamond$. Note that the subdiagram (A) below commutes by the coherence condition of the strength st.

$$\begin{tikzcd}
\underline{\mathscr{C}}(A,X) * C \arrow[rr, "\overline{[\![u_2]\!](X,s)^\diamond} * C"] \arrow[d, "\underline{\mathscr{C}}(A,X)*u_1"] & & \underline{\mathscr{C}}(B,X) * C \arrow[d, "\underline{\mathscr{C}}(B,X)*u_1"] \\
\underline{\mathscr{C}}(A,X) * TB \arrow[rr, "\overline{[\![u_2]\!](X,s)^\diamond} * TB"] \arrow[d, "\underline{\mathscr{C}}(A,X)*T(u_2)"] \arrow[dr, "\mathsf{st}_{\underline{\mathscr{C}}(A,X),B}"] & & \underline{\mathscr{C}}(B,X) * TB \arrow[d, "\mathsf{st}_{\underline{\mathscr{C}}(B,X),B}"] \\
\underline{\mathscr{C}}(A,X) * TTA \arrow[d, "\underline{\mathscr{C}}(A,X)*\mu_A"] \arrow[dr, "\mathsf{st}_{\underline{\mathscr{C}}(A,X),TA}"] & T(\underline{\mathscr{C}}(A,X) * B) \arrow[r, "T(\overline{[\![u_2]\!](X,s)^\diamond} * B)"] \arrow[d, "T(\underline{\mathscr{C}}(A,X)*u_2)"] & T(\underline{\mathscr{C}}(B,X) * B) \arrow[ddd, "T(\epsilon^B_X)"] \\
\underline{\mathscr{C}}(A,X) * TA \arrow[d, "\mathsf{st}_{\underline{\mathscr{C}}(A,X),A}"] \arrow[dr, phantom, "\text{(A)}"] & T(\underline{\mathscr{C}}(A,X) * TA) \arrow[d, "T(\mathsf{st}_{\underline{\mathscr{C}}(A,X),A})"] & \\
T(\underline{\mathscr{C}}(A,X) * A) \arrow[d, "T(\epsilon^A_X)"] & TT(\underline{\mathscr{C}}(A,X) * A) \arrow[l, "\mu_{\underline{\mathscr{C}}(A,X)*A}"'] \arrow[d, "TT(\epsilon^A_X)"] & \\
TX \arrow[dr, "s"] & TTX \arrow[l, "\mu_X"'] \arrow[r, "T(s)"] & TX \arrow[dl, "s"'] \\
& X &
\end{tikzcd}$$

Analogously, $[\![v_1\{v_2\}]\!](X,s)^\diamond$ factors as the composite $[\![v_1]\!](X,s)^\diamond \circ (\overline{[\![v_2]\!](X,s)^\diamond} * C)$, and thus it follows that $[\![u_1\{u_2\}]\!](X,s)^\diamond = [\![v_1\{v_2\}]\!](X,s)^\diamond$.

To show the soundness of the rule Ext, we assume that an $\mathbb{S}$-algebra (X, s) satisfies an equation $u \equiv v : C \to TA$; that is to say, that

$$[\![u]\!](X,s)^\diamond = [\![v]\!](X,s)^\diamond \ : \ \underline{\mathscr{C}}(A,X) * C \to X\ .$$

The interpretation map $[\![\langle V\rangle u]\!](X,s)^\diamond : \underline{\mathscr{C}}(V*A,X)*(V*C) \to X$ of the TES-term $\langle V\rangle u$ factors as the composite

$$[\![u]\!](X,s)^\diamond \circ (\overline{\mathbf{p}} * C) \circ \widetilde{\alpha}^{-1}_{\underline{\mathscr{C}}(V*A,X),V,C}\ ,$$

as shown in the diagram below, where the map $\overline{\mathbf{p}} : \underline{\mathscr{C}}(V * A, X) \cdot V \to \underline{\mathscr{C}}(A, X)$ is the transpose of the map

$$\mathbf{p} :\ \big(\underline{\mathscr{C}}(V * A, X) \cdot V\big) * A \xrightarrow{\widetilde{\alpha}_{\underline{\mathscr{C}}(V*A,X),V,A}} \underline{\mathscr{C}}(V * A, X) * (V * A) \xrightarrow{\epsilon_X^{V*A}} X\ .$$

Note that the subdiagram (A) below commutes by the coherence condition of the strength st.

$$\begin{tikzcd}
\underline{\mathscr{C}}(V*A,X)*(V*C) \arrow[r, "\widetilde{\alpha}^{-1}_{\underline{\mathscr{C}}(V*A,X),V,C}"] \arrow[d, "\underline{\mathscr{C}}(V*A,X)*(V*u)"] & (\underline{\mathscr{C}}(V*A,X)\cdot V)*C \arrow[r, "\overline{\mathbf{p}}*C"] \arrow[d, "(\underline{\mathscr{C}}(V*A,X)\cdot V)*u"] & \underline{\mathscr{C}}(A,X)*C \arrow[d, "\underline{\mathscr{C}}(A,X)*u"] \\
\underline{\mathscr{C}}(V*A,X)*(V*TA) \arrow[r, "\widetilde{\alpha}^{-1}_{\underline{\mathscr{C}}(V*A,X),V,TA}"] \arrow[d, "\underline{\mathscr{C}}(V*A,X)*\mathsf{st}_{V,A}"] & (\underline{\mathscr{C}}(V*A,X)\cdot V)*TA \arrow[r, "\overline{\mathbf{p}}*TA"] \arrow[d, "\mathsf{st}_{\underline{\mathscr{C}}(V*A,X)\cdot V,A}"] & \underline{\mathscr{C}}(A,X)*TA \arrow[dd, "\mathsf{st}_{\underline{\mathscr{C}}(A,X),A}"] \\
\underline{\mathscr{C}}(V*A,X)*T(V*A) \arrow[d, "\mathsf{st}_{\underline{\mathscr{C}}(V*A,X),V*A}"] \arrow[r, phantom, "(\mathsf{A})"] & T((\underline{\mathscr{C}}(V*A,X)\cdot V)*A) \arrow[dl, "T(\widetilde{\alpha}_{\underline{\mathscr{C}}(V*A,X),V,A})"] \arrow[d, "T(\mathbf{p})"] \arrow[dr, "T(\overline{\mathbf{p}}*A)"] & \\
T(\underline{\mathscr{C}}(V*A,X)*(V*A)) \arrow[r, "T(\epsilon_X^{V*A})"'] & TX \arrow[d, "s"] & T(\underline{\mathscr{C}}(A,X)*A) \arrow[l, "T(\epsilon_X^{A})"] \\
& X &
\end{tikzcd}$$

Analogously, $[\![\langle V\rangle v]\!](X,s)^\diamond$ factors as the composite $[\![v]\!](X,s)^\diamond \circ (\overline{\mathbf{p}} * C) \circ \widetilde{\alpha}^{-1}_{\underline{\mathscr{C}}(V*A,X),V,C}$, and thus it holds that $[\![\langle V\rangle u]\!](X,s)^\diamond = [\![\langle V\rangle v]\!](X,s)^\diamond$.

To show the soundness of the rule Local, we assume that an $\mathbb{S}$-algebra (X, s) satisfies a family of equations $\{\, u \circ e_i \equiv v \circ e_i : C_i \to TA \,\}_{i \in I}$ for TES-terms $u, v : C \to TA$ and a jointly epimorphic family of maps $\{\, e_i : C_i \to C \,\}_{i\in I}$; that is to say, that

$$\text{for all } i \in I, \quad [\![u \circ e_i]\!](X,s)^\diamond = [\![v \circ e_i]\!](X,s)^\diamond \ :\ \underline{\mathscr{C}}(A,X) * C_i \to X\ .$$

It follows from their definitions that the maps $[\![u \circ e_i]\!](X,s)^\diamond$ and $[\![v \circ e_i]\!](X,s)^\diamond$ respectively factor as the composites $[\![u]\!](X,s)^\diamond \circ (\underline{\mathscr{C}}(A,X) * e_i)$ and $[\![v]\!](X,s)^\diamond \circ (\underline{\mathscr{C}}(A,X) * e_i)$. As the action $*$ is left-closed, the functor $\underline{\mathscr{C}}(A,X) * (-)$ has a right adjoint and thus the family of maps $\{\, \underline{\mathscr{C}}(A,X) * e_i \,\}_{i\in I}$ is also jointly epimorphic. Hence, we have that $[\![u]\!](X,s)^\diamond = [\![v]\!](X,s)^\diamond$. □

7.2 Equational reasoning by rewriting

We show an internal completeness result for TESs that admit free algebras, and discuss how one may obtain a rewriting equational reasoning from it.

7.2.1 Internal completeness

Let $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ be a TES admitting free algebras, *i.e.*, such that the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint. We denote the free $\mathbb{S}$-algebra on $X \in \mathscr{C}$ as

$(T_{\mathbb{S}}X, \tau_X^{\mathbb{S}} : TT_{\mathbb{S}}X \to T_{\mathbb{S}}X)$ and the associated monad, which we call *free* $\mathbb{S}$*-algebra monad*, as $\mathbf{T}_{\mathbb{S}} = (T_{\mathbb{S}}, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$. By the universal property of the monad $\mathbf{T}$, we have a family of maps $\{\, \mathsf{q}_X^{\mathbb{S}} : TX \to T_{\mathbb{S}}X \,\}_{X\in\mathscr{C}}$ given as the unique maps such that

$$\begin{tikzcd}
TTX \arrow[r, "T(\mathsf{q}_X^{\mathbb{S}})"] \arrow[d, "\mu_X"] & TT_{\mathbb{S}}X \arrow[d, "\tau_X^{\mathbb{S}}"] \\
TX \arrow[r, "\exists!\,\mathsf{q}_X^{\mathbb{S}}"] & T_{\mathbb{S}}X \\
X \arrow[u, "\eta_X"] \arrow[ur, "\eta_X^{\mathbb{S}}"'] &
\end{tikzcd} \tag{7.2}$$

commutes. We call them *quotient maps* of the TES $\mathbb{S}$. It is easily shown that the family of maps $\{\, \tau_X^{\mathbb{S}} \,\}_{X\in\mathscr{C}}$ and $\{\, \mathsf{q}_X^{\mathbb{S}} \,\}_{X\in\mathscr{C}}$ are natural in X.

Our main result in this section, called *internal completeness*, is given in the following theorem. Note that the equivalence of the first two statements below is a form of strong completeness: it states that an equation is satisfied by all models if and only if it is satisfied by a freely generated, hence somewhat syntactic, one.

Theorem 7.2.1 (Internal completeness)**.** *For a TES* $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ *admitting free algebras, the following are equivalent:*

1. $\mathbb{S}$-$\mathbf{Alg} \models u \equiv v : C \to TA$
2. $(T_{\mathbb{S}}A, \tau_A^{\mathbb{S}}) \models u \equiv v : C \to TA$
3. $\mathsf{q}_A^{\mathbb{S}} \circ u \;=\; \mathsf{q}_A^{\mathbb{S}} \circ v : C \to T_{\mathbb{S}}A$

We introduce several lemmas before we proceed to prove the theorem. First, to apply Proposition 6.3.9 to the full subcategory $\mathbb{S}$-$\mathbf{Alg}$ of T-$\mathbf{Alg}$, we need show that $\mathbb{S}$-$\mathbf{Alg}$ is closed under the operation $[\![V, -]\!]$ in T-$\mathbf{Alg}$ for every $V \in \mathscr{V}$.

Lemma 7.2.2. *Let* $\mathbf{T} = (T, \eta, \mu, \mathsf{st})$ *be a TES-syntax on a TES-universe* $(\mathscr{C}, \mathscr{V}, *)$*. Then, for every* T*-algebra* (X, s)*, the following hold:*

1. $(X, s) \in \mathscr{C}^{\mathbf{T}} \quad \iff \quad \forall_{V\in\mathscr{V}}\ [\![V, (X, s)]\!] \in \mathscr{C}^{\mathbf{T}}$ *, and*
2. $(X, s) \models u \equiv v \quad \iff \quad \forall_{V\in\mathscr{V}}\ [\![V, (X, s)]\!] \models u \equiv v$.

Proof. Recall from Definition 6.3.8 that the structure map $[\![V, (X, s)]\!]^{\diamond} : T[\![V, X]\!] \to [\![V, X]\!]$ of the T-algebra $[\![V, (X, s)]\!]$ is given by the transpose of the composite

$$V * T[\![V, X]\!] \xrightarrow{\mathsf{st}_{V,[\![V,X]\!]}} T(V * [\![V, X]\!]) \xrightarrow{T(\epsilon_X^V)} TX \xrightarrow{\ s\ } X\ .$$

$\Rightarrow$ **part of 1.** For $(X, s) \in \mathscr{C}^{\mathbf{T}}$, the equalities

$$\begin{aligned}
[\![V, (X, s)]\!]^{\diamond} \circ \eta_{[\![V,X]\!]} &= \mathrm{id}_{[\![V,X]\!]} &&: [\![V, X]\!] \to [\![V, X]\!]\ , \\
\mu_{[\![V,X]\!]} \circ [\![V, (X, s)]\!]^{\diamond} &= T([\![V, (X, s)]\!]^{\diamond}) \circ [\![V, (X, s)]\!]^{\diamond} &&: TT[\![V, X]\!] \to [\![V, X]\!]
\end{aligned}$$

follow from the equalities between their transposes, shown in the following commutative diagrams:

$$\begin{tikzcd}
V * \llbracket V, X \rrbracket \arrow[rr, "\epsilon^V_X"] \arrow[d, "V*\eta_{\llbracket V,X \rrbracket}"'] \arrow[dr, "\eta_{V*\llbracket V,X \rrbracket}"] \arrow[d, phantom, shift left=6ex, "\mathsf{(A)}"] & & X \arrow[d, "\eta_X"] \arrow[dr, "\mathrm{id}"] & \\
V * T\llbracket V, X \rrbracket \arrow[r, "\mathsf{st}_{V,\llbracket V,X \rrbracket}"] & T(V * \llbracket V, X \rrbracket) \arrow[r, "T(\epsilon^V_X)"] & TX \arrow[r, "s"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
V * TT\llbracket V, X \rrbracket \arrow[r, "V*\mu_{\llbracket V,X \rrbracket}"] \arrow[dr, "\mathsf{st}_{V,T\llbracket V,X \rrbracket}"] \arrow[dd, "V*T(\llbracket V,(X,s) \rrbracket^\diamond)"] & V * T\llbracket V, X \rrbracket \arrow[r, "\mathsf{st}_{V,\llbracket V,X \rrbracket}"] \arrow[d, phantom, "\mathsf{(B)}"] & T(V * \llbracket V, X \rrbracket) \arrow[r, "T(\epsilon^V_X)"] & TX \arrow[dr, "s"] & \\
& T(V * T\llbracket V, X \rrbracket) \arrow[r, "T(\mathsf{st}_{V,\llbracket V,X \rrbracket})"] \arrow[d, "T(V*\llbracket V,(X,s) \rrbracket^\diamond)"] & TT(V * \llbracket V, X \rrbracket) \arrow[r, "TT(\epsilon^V_X)"] \arrow[u, "\mu_{V*\llbracket V,X \rrbracket}"] & TTX \arrow[u, "\mu_X"'] \arrow[d, "T(s)"'] & X \\
V * T\llbracket V, X \rrbracket \arrow[r, "\mathsf{st}_{V,\llbracket V,X \rrbracket}"'] & T(V * \llbracket V, X \rrbracket) \arrow[rr, "T(\epsilon^V_X)"'] & & TX \arrow[ur, "s"'] &
\end{tikzcd}$$

where the diagrams $\mathsf{(A)}$ and $\mathsf{(B)}$ commute by the coherence condition of the strength st.

$\Rightarrow$ **part of 2.** Let (X, s) be a T-algebra satisfying an equation $(X, s) \models u \equiv v : C \to TA$, *i.e.*, such that $\llbracket u \rrbracket(X, s)^\diamond = \llbracket v \rrbracket(X, s)^\diamond : \underline{\mathscr{C}}(A, X) * C \to X$. From the commutative diagram below, it follows that the transpose

$$\overline{\llbracket u \rrbracket(\llbracket V, (X, s) \rrbracket)^\diamond} : V * (\underline{\mathscr{C}}(A, \llbracket V, X \rrbracket) * C) \to X$$

of the interpretation map $\llbracket u \rrbracket(\llbracket V, (X, s) \rrbracket)^\diamond : \underline{\mathscr{C}}(A, \llbracket V, X \rrbracket) * C \to \llbracket V, X \rrbracket$ factors as the composite

$$\llbracket u \rrbracket(X, s)^\diamond \circ (\overline{\mathbf{p}} * C) \circ \widetilde{\alpha}^{-1}_{V,\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket),C}$$

where the map $\overline{\mathbf{p}} : V \cdot \underline{\mathscr{C}}(A, \llbracket V, X \rrbracket) \to \underline{\mathscr{C}}(A, X)$ is the transpose of the map

$$\mathbf{p} :\ (V \cdot \underline{\mathscr{C}}(A, \llbracket V, X \rrbracket)) * A \xrightarrow{\widetilde{\alpha}_{V,\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket),A}} V * (\underline{\mathscr{C}}(A, \llbracket V, X \rrbracket) * A) \xrightarrow{V*\epsilon^A_{\llbracket V,X \rrbracket}} V * \llbracket V, X \rrbracket \xrightarrow{\epsilon^V_X} X\,.$$

Note that the subdiagram $\mathsf{(A)}$ below commute by the coherence condition of the strength st.

$$\begin{tikzcd}
V * (\underline{\mathscr{C}}(A, \llbracket V, X \rrbracket) * C) \arrow[rr, "\widetilde{\alpha}^{-1}_{V,\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket),C}"] \arrow[d, "V*(\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket)*u)"] & & (V \cdot \underline{\mathscr{C}}(A, \llbracket V, X \rrbracket)) * C \arrow[r, "\overline{\mathbf{p}}*C"] \arrow[d, "(V\cdot\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket))*u"] & \underline{\mathscr{C}}(A, X) * C \arrow[d, "\underline{\mathscr{C}}(A,X)*u"] \\
V * (\underline{\mathscr{C}}(A, \llbracket V, X \rrbracket) * TA) \arrow[rr, "\widetilde{\alpha}^{-1}_{V,\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket),TA}"] \arrow[d, "V*\mathsf{st}_{\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket),A}"] \arrow[drr, phantom, "\mathsf{(A)}"] & & (V \cdot \underline{\mathscr{C}}(A, \llbracket V, X \rrbracket)) * TA \arrow[r, "\overline{\mathbf{p}}*TA"] \arrow[d, "\mathsf{st}_{V\cdot\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket),A}"] & \underline{\mathscr{C}}(A, X) * TA \arrow[ddd, "\mathsf{st}_{\underline{\mathscr{C}}(A,X),A}"] \\
V * T(\underline{\mathscr{C}}(A, \llbracket V, X \rrbracket) * A) \arrow[dr, "\mathsf{st}_{V,\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket)*A}"] \arrow[dd, "V*T(\epsilon^A_{\llbracket V,X \rrbracket})"'] & & T((V \cdot \underline{\mathscr{C}}(A, \llbracket V, X \rrbracket)) * A) \arrow[dl, "T(\widetilde{\alpha}_{V,\underline{\mathscr{C}}(A,\llbracket V,X \rrbracket),A})"'] \arrow[dd, "T(\mathbf{p})"] \arrow[ddr, "T(\overline{\mathbf{p}}*A)"] & \\
& T(V * (\underline{\mathscr{C}}(A, \llbracket V, X \rrbracket) * A)) \arrow[d, "T(V*\epsilon^A_{\llbracket V,X \rrbracket})"] & & \\
V * T\llbracket V, X \rrbracket \arrow[r, "\mathsf{st}_{V,\llbracket V,X \rrbracket}"'] & T(V * \llbracket V, X \rrbracket) \arrow[r, "T(\epsilon^V_X)"'] & TX \arrow[d, "s"] & T(\underline{\mathscr{C}}(A, X) * A) \arrow[l, "T(\epsilon^A_X)"] \\
& & X &
\end{tikzcd}$$

Analogously, $\overline{[\![v]\!]([\![V,(X,s)]\!])^\diamond}$ factors as the composite $[\![v]\!](X,s)^\diamond \circ (\overline{\mathbf{p}} * C) \circ \widetilde{\alpha}^{-1}_{V,\underline{\mathscr{C}}(A,[\![V,X]\!]),C}$ and hence $[\![u]\!]([\![V,(X,s)]\!])^\diamond = [\![v]\!]([\![V,(X,s)]\!])^\diamond$.

$\Leftarrow$ **part of 1 and 2.** For any T-algebra (X,s), one can easily show that the canonical isomorphism $X \cong [\![I,X]\!]$ constitutes an isomorphism between the T-algebras (X,s) and $[\![I,(X,s)]\!]$. As the laws for Eilenberg-Moore algebras and the TES-equation $u \equiv v$ can be expressed as functorial equations on T-**Alg**, it follows, by Proposition 3.3.2, that the categories $\mathscr{C}^{\mathbf{T}}$ and $T\text{-}\mathbf{Alg}/_{u\equiv v}$ are isomorphism-closed subcategories of T-**Alg**, for $T\text{-}\mathbf{Alg}/_{u\equiv v}$ the full subcategory of T-**Alg** consisting of T-algebras satisfying $u \equiv v$. Thus, if the T-algebra $[\![I,(X,s)]\!]$ is in $\mathscr{C}^{\mathbf{T}}$ (resp. in $T\text{-}\mathbf{Alg}/_{u\equiv v}$), then so is the T-algebra (X,s), as (X,s) is isomorphic to $[\![I,(X,s)]\!]$ in T-**Alg**. $\square$

By Proposition 6.3.9 (2), we have that the free $\mathbb{S}$-algebra monad $\mathbf{T}_{\mathbb{S}}$ is strong, with the components of the strength $\mathsf{st}^{\mathbb{S}}$ given by the unique maps such that the following diagram commutes:

$$\begin{tikzcd}
V * TT_{\mathbb{S}}X \arrow[r, "\mathsf{st}_{V,T_{\mathbb{S}}X}"] \arrow[d, "V*\tau^{\mathbb{S}}_X"'] & T(V * T_{\mathbb{S}}X) \arrow[r, "T(\mathsf{st}^{\mathbb{S}}_{V,X})"] & TT_{\mathbb{S}}(V * X) \arrow[d, "\tau^{\mathbb{S}}_{V*X}"] \\
V * T_{\mathbb{S}}X \arrow[rr, dashed, "\exists!\, \mathsf{st}^{\mathbb{S}}_{V,X}"] & & T_{\mathbb{S}}(V * X) \\
V * X \arrow[u, "V*\eta^{\mathbb{S}}_X"] \arrow[urr, "\eta^{\mathbb{S}}_{V*X}"'] & &
\end{tikzcd}$$

We now show that the natural transformation $\mathsf{q}^{\mathbb{S}} : T \dot{\to} T_{\mathbb{S}}$ is a strong functor morphism.

Proposition 7.2.3. *Let $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ be a TES admitting free algebras. Then, the natural transformation $\mathsf{q}^{\mathbb{S}} : T \dot{\to} T_{\mathbb{S}}$ is a strong functor morphism between the strong monads $\mathbf{T}$ and $\mathbf{T}_{\mathbb{S}}$; that is, the following diagram commutes:*

$$\begin{tikzcd}
V * TX \arrow[r, "V*\mathsf{q}^{\mathbb{S}}_X"] \arrow[d, "\mathsf{st}_{V,X}"'] & V * T_{\mathbb{S}}X \arrow[d, "\mathsf{st}^{\mathbb{S}}_{V,X}"] \\
T(V * X) \arrow[r, "\mathsf{q}^{\mathbb{S}}_{V*X}"'] & T_{\mathbb{S}}(V * X)
\end{tikzcd}$$

Proof. The commutativity of the above diagram follows from Proposition 6.3.9 (1) and the commutativity of the following two diagrams:

$$\begin{tikzcd}
V * TTX \arrow[r, "\mathsf{st}_{V,TX}"] \arrow[d, "V*\mu_X"'] & T(V * TX) \arrow[r, "T(\mathsf{st}_{V,X})"] & TT(V * X) \arrow[r, "T(\mathsf{q}^{\mathbb{S}}_{V*X})"] \arrow[d, "\mu_{V*X}"] & TT_{\mathbb{S}}(V * X) \arrow[d, "\tau^{\mathbb{S}}_{V*X}"] \\
V * TX \arrow[rr, "\mathsf{st}_{V,X}"] & & T(V * X) \arrow[r, "\mathsf{q}^{\mathbb{S}}_{V*X}"] & T_{\mathbb{S}}(V * X) \\
V * X \arrow[u, "V*\eta_X"] \arrow[urr, "\eta_{V*X}"] \arrow[urrr, "\eta^{\mathbb{S}}_{V*X}"'] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
V * TTX \arrow[r, "\mathsf{st}_{V,TX}"] \arrow[dd, "V*\mu_X"'] \arrow[dr, "V*T(\mathsf{q}^{\mathbb{S}}_X)"'] & T(V * TX) \arrow[r, "T(V*\mathsf{q}^{\mathbb{S}}_X)"] & T(V * T_{\mathbb{S}}X) \arrow[r, "T(\mathsf{st}^{\mathbb{S}}_{V,X})"] & TT_{\mathbb{S}}(V * X) \arrow[dd, "\tau^{\mathbb{S}}_{V*X}"] \\
& V * TT_{\mathbb{S}}X \arrow[ur, "\mathsf{st}_{V,T_{\mathbb{S}}X}"'] \arrow[d, "V*\tau^{\mathbb{S}}_X"] & & \\
V * TX \arrow[r, "V*\mathsf{q}^{\mathbb{S}}_X"] & V * T_{\mathbb{S}}X \arrow[rr, "\mathsf{st}^{\mathbb{S}}_{V,X}"] & & T_{\mathbb{S}}(V * X) \\
V * X \arrow[u, "V*\eta_X"] \arrow[ur, "V*\eta^{\mathbb{S}}_X"] \arrow[urrr, "\eta^{\mathbb{S}}_{V*X}"'] & & &
\end{tikzcd}$$

□

We are finally ready to prove the internal completeness theorem.

Proof of Theorem 7.2.1. We show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

Proof of 1 $\Rightarrow$ 2. It holds vacuously.

Proof of 2 $\Rightarrow$ 3. The map $\mathsf{q}^{\mathbb{S}}_A \circ u : C \to T_{\mathbb{S}}A$ factors as the composite

$$[\![u]\!](T_{\mathbb{S}}A, \tau^{\mathbb{S}}_A)^{\diamond} \circ (\overline{\mathbf{p}} * C) \circ \widetilde{\lambda}_C^{-1} \,,$$

as shown by the commutative diagram below, where the map $\overline{\mathbf{p}} : I \to \underline{\mathscr{C}}(A, T_{\mathbb{S}}A)$ is the transpose of the map $\mathbf{p} : I * A \xrightarrow{\widetilde{\lambda}_A} A \xrightarrow{\eta^{\mathbb{S}}_A} T_{\mathbb{S}}A$.

$$\begin{tikzcd}
C \arrow[rr, "u"] \arrow[d, "\widetilde{\lambda}_C^{-1}"] & & TA \arrow[r, "\mathsf{q}^{\mathbb{S}}_A"] \arrow[dr, "T(\eta^{\mathbb{S}}_A)"'] \arrow[dr, phantom, shift left=6, "(\mathsf{B})"] & T_{\mathbb{S}}A \\
I * C \arrow[r, "I*u"] \arrow[dr, "\overline{\mathbf{p}}*C"'] & I * TA \arrow[ur, "\widetilde{\lambda}_{TA}"] \arrow[r, "\mathsf{st}_{I,A}"] \arrow[dr, "\overline{\mathbf{p}}*TA"] \arrow[ur, phantom, shift right=4, "(\mathsf{A})"] & T(I * A) \arrow[u, "T(\widetilde{\lambda}_A)"] \arrow[dr, "T(\overline{\mathbf{p}}*A)"] & TT_{\mathbb{S}}A \arrow[u, "\tau^{\mathbb{S}}_A"'] \\
& \underline{\mathscr{C}}(A, T_{\mathbb{S}}A) * C \arrow[r, "\underline{\mathscr{C}}(A,T_{\mathbb{S}}A)*u"'] & \underline{\mathscr{C}}(A, T_{\mathbb{S}}A) * TA \arrow[r, "\mathsf{st}_{\underline{\mathscr{C}}(A,T_{\mathbb{S}}A),A}"'] & T(\underline{\mathscr{C}}(A, T_{\mathbb{S}}A) * A) \arrow[u, "T(\epsilon^A_{T_{\mathbb{S}}A})"']
\end{tikzcd}$$

Note that the subdiagram (A) commutes by the coherence condition of the strength st, and the commutativity of (B) follows from the two commutative diagrams below, by the universal property of the monad $\mathbf{T}$:

$$\begin{tikzcd}
TTA \arrow[r, "T(\mathsf{q}^{\mathbb{S}}_A)"] \arrow[d, "\mu_A"] & TT_{\mathbb{S}}A \arrow[d, "\tau^{\mathbb{S}}_A"] \\
TA \arrow[r, "\mathsf{q}^{\mathbb{S}}_A"] & T_{\mathbb{S}}A \\
A \arrow[u, "\eta_A"] \arrow[ur, "\eta^{\mathbb{S}}_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTA \arrow[r, "TT(\eta^{\mathbb{S}}_A)"] \arrow[d, "\mu_A"] & TTT_{\mathbb{S}}A \arrow[r, "T(\tau^{\mathbb{S}}_A)"] \arrow[d, "\mu_{T_{\mathbb{S}}A}"] & TT_{\mathbb{S}}A \arrow[d, "\tau^{\mathbb{S}}_A"] \\
TA \arrow[r, "T(\eta^{\mathbb{S}}_A)"] & TT_{\mathbb{S}}A \arrow[r, "\tau^{\mathbb{S}}_A"] & T_{\mathbb{S}}A \\
A \arrow[u, "\eta_A"] \arrow[r, "\eta^{\mathbb{S}}_A"] & T_{\mathbb{S}}A \arrow[u, "\eta_{T_{\mathbb{S}}A}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

where the right half of the right diagram commutes because $(T_{\mathbb{S}}A, \tau^{\mathbb{S}}_A)$ is an Eilenberg-Moore algebra for $\mathbf{T}$.

Analogously, the map $\mathsf{q}^{\mathbb{S}}_A \circ v$ factors as the composite $[\![v]\!](T_{\mathbb{S}}A, \tau^{\mathbb{S}}_A)^{\diamond} \circ (\overline{\mathbf{p}} * C) \circ \widetilde{\lambda}_C^{-1}$ and thus it follows that $[\![u]\!](T_{\mathbb{S}}A, \tau^{\mathbb{S}}_A)^{\diamond} = [\![v]\!](T_{\mathbb{S}}A, \tau^{\mathbb{S}}_A)^{\diamond}$ implies $\mathsf{q}^{\mathbb{S}}_A \circ u = \mathsf{q}^{\mathbb{S}}_A \circ v$.

Proof of 3 $\Rightarrow$ 1. For any $(X, s : TX \to X) \in \mathbb{S}$-**Alg**, the interpretation map $[\![u]\!](X,s)^{\diamond} : \underline{\mathscr{C}}(A,X) * C \to X$ factors as the composite

$$s^* \circ T_{\mathbb{S}}(\epsilon^A_X) \circ \mathsf{st}^{\mathbb{S}}_{\underline{\mathscr{C}}(A,X),A} \circ \big(\underline{\mathscr{C}}(A,X) * (\mathsf{q}^{\mathbb{S}}_A \circ u)\big)\,,$$

as shown in the commutative diagram below, where $s^* : T_{\mathbb{S}}X \to X$ denotes the unique map such that

$$\begin{tikzcd}
TT_{\mathbb{S}}X \arrow[r, "T(s^*)"] \arrow[d, "\tau^{\mathbb{S}}_X"'] & TX \arrow[d, "s"] \\
T_{\mathbb{S}}X \arrow[r, "s^*"] & X \\
X \arrow[u, "\eta^{\mathbb{S}}_X"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

commutes.

$$\begin{tikzcd}
\underline{\mathscr{C}}(A,X) * C \arrow[r, "\underline{\mathscr{C}}(A,X)*u"] & \underline{\mathscr{C}}(A,X) * TA \arrow[r, "\mathsf{st}_{\underline{\mathscr{C}}(A,X),A}"] \arrow[d, "\underline{\mathscr{C}}(A,X)*\mathsf{q}^{\mathbb{S}}_A"'] \arrow[dr, phantom, "(\mathsf{A})"] & T(\underline{\mathscr{C}}(A,X) * A) \arrow[r, "T(\epsilon^A_X)"] \arrow[d, "\mathsf{q}^{\mathbb{S}}_{\underline{\mathscr{C}}(A,X)*A}"] & TX \arrow[r, "s"] \arrow[d, "\mathsf{q}^{\mathbb{S}}_X"'] \arrow[dr, phantom, "(\mathsf{B})"] & X \\
 & \underline{\mathscr{C}}(A,X) * T_{\mathbb{S}}A \arrow[r, "\mathsf{st}^{\mathbb{S}}_{\underline{\mathscr{C}}(A,X),A}"] & T_{\mathbb{S}}(\underline{\mathscr{C}}(A,X) * A) \arrow[r, "T_{\mathbb{S}}(\epsilon^A_X)"] & T_{\mathbb{S}}X \arrow[ur, "s^*"'] &
\end{tikzcd}$$

The subdiagram (A) above commutes because $\mathsf{q}^{\mathbb{S}}$ is a strong functor morphism from $\mathbf{T}$ to $\mathbf{T}_{\mathbb{S}}$, and the commutativity of (B) follows from the commutativity of the two diagrams below, by the universal property of the monad $\mathbf{T}$:

$$\begin{tikzcd}
TTX \arrow[r, "T(\mathsf{q}^{\mathbb{S}}_X)"] \arrow[d, "\mu_X"'] & TT_{\mathbb{S}}X \arrow[r, "T(s^*)"] \arrow[d, "\tau^{\mathbb{S}}_X"] & TX \arrow[d, "s"] \\
TX \arrow[r, "\mathsf{q}^{\mathbb{S}}_X"] & T_{\mathbb{S}}X \arrow[r, "s^*"] & X \\
X \arrow[u, "\eta_X"] \arrow[ur, "\eta^{\mathbb{S}}_X"] \arrow[urr, "\mathrm{id}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTX \arrow[r, "T(s)"] \arrow[d, "\mu_X"'] & TX \arrow[d, "s"] \\
TX \arrow[r, "s"] & X \\
X \arrow[u, "\eta_X"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

where the right diagram commutes because (X, s) is an Eilenberg-Moore algebra for $\mathbf{T}$.

Analogously, the map $[\![v]\!](X,s)^{\diamond}$ factors as the composite

$$s^* \circ T_{\mathbb{S}}(\epsilon^A_X) \circ \mathsf{st}^{\mathbb{S}}_{\underline{\mathscr{C}}(A,X),A} \circ (\underline{\mathscr{C}}(A,X) * (\mathsf{q}^{\mathbb{S}}_A \circ v))$$

and thus it follows that $\mathsf{q}^{\mathbb{S}}_A \circ u = \mathsf{q}^{\mathbb{S}}_A \circ v$ implies $[\![u]\!](X,s)^{\diamond} = [\![v]\!](X,s)^{\diamond}$. $\square$

7.2.2 Towards complete reasoning by rewriting

We give conditions under which TESs admit free algebras, and show how quotient maps of TESs are constructed using the theory of equational systems developed in Part I. Then we see by an example how the construction might yield a sound and complete rewriting equational reasoning.

Recall from Section 6.4 that, for a category $\mathscr{C}$ with small coproducts, every TES $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ induces the monadic equational system $\overline{\mathbb{S}} = (\mathscr{C} : \mathbf{T} \rhd \Gamma_E \vdash \overline{L} \equiv \overline{R})$ such that $\mathbb{S}$-$\mathbf{Alg} = \overline{\mathbb{S}}$-$\mathbf{Alg}$, where

$$\Gamma_E(X) = \textstyle\coprod_{(u \,\equiv\, v : C \to TA) \in E} \underline{\mathscr{C}}(A, X) * C \ ,$$
$$\overline{L}(X, s) = \big[\, [\![u]\!](X, s)^{\diamond} \,\big]_{(u \,\equiv\, v) \in E} \ , \qquad \overline{R}(X, s) = \big[\, [\![v]\!](X, s)^{\diamond} \,\big]_{(u \,\equiv\, v) \in E} \ .$$

Recalling the notions of finitary and inductive monadic equational systems from Definition 4.1.14, we define notions of compactness and projectiveness for objects, and notions of finitary and inductive TES.

Definition 7.2.4. Let $(\mathscr{C}, \mathscr{V}, *)$ be a TES-universe. An object A in $\mathscr{C}$ is respectively said to be *κ-compact*, for κ an infinite limit ordinal, and *projective* if the functor $\underline{\mathscr{C}}(A, -)$ from $\mathscr{C}$ to $\mathscr{V}$ is respectively κ-cocontinuous and epicontinuous.

Definition 7.2.5. A TES $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ is called *κ-finitary*, for κ an infinite limit ordinal, if the category $\mathscr{C}$ is cocomplete, the endofunctor T on $\mathscr{C}$ is κ-cocontinuous, and the arity A of each equation $u \equiv v : C \to TA$ in E is κ-compact. Such a TES is called *κ-inductive* if furthermore the endofunctor T is epicontinuous and the arity A of each equation $u \equiv v : C \to TA$ in E is projective.

The following theorem is a direct consequence of the above definitions.

Theorem 7.2.6. *For a TES $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$, if $\mathbb{S}$ is κ-finitary for some infinite limit ordinal κ, then the associated monadic equational system $\overline{\mathbb{S}}$ is κ-finitary and thus the TES $\mathbb{S}$ admits free algebras. If furthermore $\mathbb{S}$ is κ-inductive, then $\overline{\mathbb{S}}$ is also κ-inductive.*

Proof. As the $\mathscr{V}$-action $(\mathscr{C}, *)$ is right-closed, the functor $(-) * C : \mathscr{V} \to \mathscr{C}$, for any object $C \in \mathscr{C}$, preserves colimits and, in particular, epimorphisms. Thus, for any TES-term $u \equiv v : C \to TA$ with A κ-compact (and projective), the endofunctor $\underline{\mathscr{C}}(A, -) * C$ on $\mathscr{C}$ is κ-cocontinuous (and epicontinuous). $\square$

For a κ-finitary TES $\mathbb{S} = (\mathscr{C}, \mathscr{V}, *, \mathbf{T}, E)$ with κ an infinite limit ordinal, we have the following situation:

$$\begin{tikzcd}
\mathbb{S}\text{-}\mathbf{Alg} = \overline{\mathbb{S}}\text{-}\mathbf{Alg} \arrow[r, hook, "J"'] \arrow[r, phantom, "\perp"] & \mathscr{C}^{\mathbf{T}} \arrow[l, bend right, "K"'] \arrow[d, "U_{\mathbf{T}}"] \arrow[d, phantom, "\dashv", shift right=2ex] \\
 & \mathscr{C} \arrow[u, bend left]
\end{tikzcd}$$

For each object $X \in \mathscr{C}$, since (TX, μ_X) is a free Eilenberg-Moore algebra on X, the free $\mathbb{S}$-algebra $(T_{\mathbb{S}}X, \tau^{\mathbb{S}}_X)$ on X is given by the free $\overline{\mathbb{S}}$-algebra $K(TX, \mu_X)$ over the Eilenberg-Moore algebra (TX, μ_X). Satisfying the commutative diagram (7.2), the universal homomorphism $(TX, \mu_X) \to (T_{\mathbb{S}}X, \tau^{\mathbb{S}}_X)$ induced from the adjunction $K \dashv J$ yields the quotient map $\mathsf{q}_X^{\mathbb{S}} : TX \to T_{\mathbb{S}}X$. The theory of equational systems studied in Part I presents an explicit categorical construction of the reflection $K : \mathscr{C}^{\mathbf{T}} \to \overline{\mathbb{S}}\text{-}\mathbf{Alg}$; in particular, that of the universal homomorphism $\mathsf{q}_X^{\mathbb{S}}$.

In the simple case of ω-inductive TESs, the quotient maps $\mathsf{q}_X^{\mathbb{S}}$ are constructed, by Corollary 3.2.12 and Theorem 3.2.4, as follows:

$$
\begin{tikzcd}
\forall\, (u \equiv v : C \to TA) \in E & T(TX) \arrow[r, two heads, "T(q_0)"] \arrow[dd, "\mu_X"'] \arrow[ddr, "p_0"] & T(TX)_1 \arrow[r, two heads, "T(q_1)"] \arrow[dd, phantom, "\mathsf{po}"] \arrow[ddr, "p_1"] & T(TX)_2 \arrow[r, two heads, "T(q_2)"] \arrow[dd, phantom, "\mathsf{po}"] \arrow[ddr, "p_2"] & T(TX)_3 & \cdots\cdots & T(T_{\mathbb{S}}X) \arrow[dd, dashed, "\tau_X^{\mathbb{S}}"] \\
\vdots & & & & & & \\
\underline{\mathscr{C}}(A, TX) * C \arrow[r, shift left, "{[\![u]\!](TX,\mu_X)^{\diamond}}"] \arrow[r, shift right, "{[\![v]\!](TX,\mu_X)^{\diamond}}"'] & TX \arrow[r, two heads, "q_0", "\mathsf{coeq}"'] \arrow[rrrrr, two heads, bend right, "\mathsf{q}_X^{\mathbb{S}}"'] & (TX)_1 \arrow[r, two heads, "q_1"] & (TX)_2 \arrow[r, two heads, "q_2"] & (TX)_3 & \cdots\cdots & T_{\mathbb{S}}X \, \mathsf{colim} \\
\vdots & & & & & &
\end{tikzcd}
\tag{7.3}
$$

where q_0 is the universal map that coequalizes every pair $[\![u]\!](TX, \mu_X)^{\diamond}$ and $[\![v]\!](TX, \mu_X)^{\diamond}$ with $(u \equiv v) \in E$.

Furthermore, when the strong monad $\mathbf{T}$ arises from free algebras for a strong endofunctor F which is ω-cocontinuous and epicontinuous, as in Proposition 6.3.4, the TES $\mathbb{S}$ can be encoded as the equational system $\widehat{\mathbb{S}}$ given in Section 6.4 and thus the construction of the quotient maps $\mathsf{q}_X^{\mathbb{S}}$ simplifies as follows:

$$
\begin{tikzcd}
\forall\, (u \equiv v : C \to TA) \in E & F(TX) \arrow[r, two heads, "F(q_0)"] \arrow[dd, "\widehat{\mu}_X"'] \arrow[ddr, "p_0"] & F(TX)_1 \arrow[r, two heads, "F(q_1)"] \arrow[dd, phantom, "\mathsf{po}"] \arrow[ddr, "p_1"] & F(TX)_2 \arrow[r, two heads, "F(q_2)"] \arrow[dd, phantom, "\mathsf{po}"] \arrow[ddr, "p_2"] & F(TX)_3 & \cdots\cdots & F(T_{\mathbb{S}}X) \arrow[dd, dashed, "\widehat{\tau}_X^{\mathbb{S}}"] \\
\vdots & & & & & & \\
\underline{\mathscr{C}}(A, TX) * C \arrow[r, shift left, "{[\![u]\!](TX,\mu_X)^{\diamond}}"] \arrow[r, shift right, "{[\![v]\!](TX,\mu_X)^{\diamond}}"'] & TX \arrow[r, two heads, "q_0", "\mathsf{coeq}"'] \arrow[rrrrr, two heads, bend right, "\mathsf{q}_X^{\mathbb{S}}"'] & (TX)_1 \arrow[r, two heads, "q_1"] & (TX)_2 \arrow[r, two heads, "q_2"] & (TX)_3 & \cdots\cdots & T_{\mathbb{S}}X \, \mathsf{colim} \\
\vdots & & & & & &
\end{tikzcd}
\tag{7.4}
$$

where $(TX, \widehat{\mu}_X)$ and $(T_{\mathbb{S}}X, \widehat{\tau}_X^{\mathbb{S}})$ are the F-algebras respectively corresponding to the Eilenberg-Moore algebras (TX, μ_X) and $(T_{\mathbb{S}}X, \tau_X^{\mathbb{S}})$ for the monad $\mathbf{T}$.

As we have seen in Part I, the intuition behind the above construction is that the object $T_{\mathbb{S}}X$ is obtained by quotienting the object TX by the equations in E and congruence rules for operators. Following this intuition, as exemplified in the example below and the applications in the next chapter, one may synthesize a sound and complete rewriting equational reasoning by analyzing the above construction of quotient maps and by the internal completeness.

Example 7.2.7 (continued)**.** For an algebraic theory $\mathbb{T} = (\Sigma, E)$, recall, from Example 6.3.7, that the theory $\mathbb{T}$ is encoded as the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle = (\mathbf{Set}, \mathbf{Set}, \times, \mathbf{T}_\Sigma, \langle\!\langle E \rangle\!\rangle)$ preserving models; and that the strong monad $\mathbf{T}_\Sigma$ arises from free algebras for the strong endofunctor F_Σ given by $F_\Sigma(X) = \coprod_{\mathsf{o} \in \Sigma} X^{|\mathsf{o}|}$. As the endofunctor $(-)^A$ on $\mathbf{Set}$ for every finite set A is finitary and epicontinuous, it follows that the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$ is ω-inductive.

We consider the construction (7.4) for the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$. First, we observe that the map $q_0 : T_\Sigma X \twoheadrightarrow (T_\Sigma X)_1$ is the universal map in $\mathbf{Set}$ that coequalizes every pair $[\![t]\!], [\![t']\!] : (T_\Sigma X)^V \to T_\Sigma X$ with $(V \vdash t \equiv t') \in E$, where $[\![t]\!], [\![t']\!]$ are the maps respectively sending $\{\, s_v \in T_\Sigma X \,\}_{v \in V}$ to the terms $t\{v \mapsto s_v\}$ and $t'\{v \mapsto s_v\}$ respectively obtained by simultaneously substituting s_v for each variable $v \in V$ in the terms t and t'. From this observation, it follows that the set $(T_\Sigma X)_1$ is given as the quotient set $T_\Sigma X/_{\approx_1}$ of $T_\Sigma X$ under the equivalence relation $\approx_1$ generated by the following rule:

$$\frac{}{t\{v \mapsto s_v\}_{v\in V} \approx_1 t'\{v \mapsto s_v\}_{v\in V}} \begin{array}{l}(V \vdash t \equiv t') \in E,\\ \{\, s_v \,\}_{v\in V} \in (T_\Sigma X)^V\end{array}$$

The map q_0 sends a term $s \in T_\Sigma X$ to its equivalence class $[s]_{\approx_1} \in T_\Sigma X/_{\approx_1}$, and the map p_0 sends $\iota_{\mathsf{o}}(s_1, \ldots, s_{|\mathsf{o}|}) \in F_\Sigma(T_\Sigma X)$ to $[\mathsf{o}(s_1, \ldots, s_{|\mathsf{o}|})]_{\approx_1} \in T_\Sigma X/_{\approx_1}$.

We note that a pushout of a surjective map $e : A \twoheadrightarrow B$ and a map $f : A \to C$ in **Set** is given by the cospan $f' : B \to C/_{\approx} \twoheadleftarrow C : e'$

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[dr, "f"'] & B \arrow[dr, "f'"] & \\ & C \arrow[r, two heads, "e'"] & C/_{\approx} \end{tikzcd}$$

where $C/_{\approx}$ is the quotient set of the set C under the equivalence relation $\approx$ generated by the rule $f(a) \approx f(a')$ in C for all $a, a' \in A$ such that $e(a) = e(a')$ in B; and where the surjective map $e' : C \twoheadrightarrow C/_{\approx}$ sends an element c to its equivalence class $[c]_{\approx}$, and the map $f' : B \to C/_{\approx}$ sends an element b to $e'(f(\widetilde{b}))$ for $\widetilde{b}$ an element of A such that $e(\widetilde{b}) = b$. From this observation, by inductively analyzing the construction of the maps q_n for $n \geq 1$, we have that the sets $(T_\Sigma X)_n$ for $n \geq 2$ are given as the quotient sets $T_\Sigma X/_{\approx_n}$ of $T_\Sigma X$ under the equivalence relations $\approx_n$ inductively generated by the following rules:

$$\frac{s \approx_{n-1} s'}{s \approx_n s'} \qquad \frac{s_1 \approx_{n-1} s'_1, \ \ldots, \ s_{|\mathsf{o}|} \approx_{n-1} s'_{|\mathsf{o}|}}{\mathsf{o}(s_1, \ldots, s_{|\mathsf{o}|}) \approx_n \mathsf{o}(s'_1, \ldots, s'_{|\mathsf{o}|})} \ \mathsf{o} \in \Sigma$$

The maps q_n for $n \geq 1$ send $[s]_{\approx_n} \in T_\Sigma X/_{\approx_n}$ to $[s]_{\approx_{n+1}} \in T_\Sigma X/_{\approx_{n+1}}$, and the maps p_n for $n \geq 1$ send $\iota_{\mathsf{o}}([s_1]_{\approx_n}, \ldots, [s_{|\mathsf{o}|}]_{\approx_n}) \in F_\Sigma(T_\Sigma X/_{\approx_n})$ to $[\mathsf{o}(s_1, \ldots, s_{|\mathsf{o}|})]_{\approx_{n+1}} \in T_\Sigma X/_{\approx_{n+1}}$.

By taking the colimit of the chain of quotients $\{\, q_n : T_\Sigma X/_{\approx_n} \twoheadrightarrow T_\Sigma X/_{\approx_{n+1}} \,\}_{n\geq 0}$ in **Set**, the set $T_{\mathbb{S}} X$ is given by the quotient set $T_\Sigma X/_{\approx_E}$ of $T_\Sigma X$ under the relation $\approx_E$ generated by the following rules:

$$\mathsf{Ref}\ \frac{}{s \approx_E s}\ s \in T_\Sigma X \qquad \mathsf{Sym}\ \frac{s \approx_E s'}{s' \approx_E s} \qquad \mathsf{Trans}\ \frac{s \approx_E s' \quad s' \approx_E s''}{s \approx_E s''}$$

$$\mathsf{Axiom}\ \frac{}{t\{v \mapsto s_v\}_{v\in V} \approx_E t'\{v \mapsto s_v\}_{v\in V}} \begin{array}{l}(V \vdash t \equiv t') \in E,\\ \{\, s_v \,\}_{v\in V} \in (T_\Sigma X)^V\end{array} \tag{7.5}$$

$$\mathsf{Cong}\ \frac{s_1 \approx_E s'_1, \ \ldots, \ s_{|\mathsf{o}|} \approx_E s'_{|\mathsf{o}|}}{\mathsf{o}(s_1, \ldots, s_{|\mathsf{o}|}) \approx_E \mathsf{o}(s'_1, \ldots, s'_{|\mathsf{o}|})} \ \mathsf{o} \in \Sigma$$

The quotient map $\mathsf{q}^{\mathbb{S}}_X : T_\Sigma X \to T_{\mathbb{S}} X$ sends a term s to its equivalence class $[s]_{\approx_E}$.

Remark 7.2.8. The analysis of the construction (7.3) for the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$ yields as the set $T_{\mathbb{S}} X$ the quotient set $T_\Sigma X/_{\approx'_E}$ for $\approx'_E$ the equivalence relation on $T_\Sigma X$ generated by the same rules as $\approx_E$ except that the rule Cong is replaced by the following rule Cong':

$$\mathsf{Cong}'\ \frac{s_1 \approx_E s'_1, \ \ldots, \ s_k \approx_E s'_k}{\mathbf{C}[s_1, \ldots, s_k] \approx_E \mathbf{C}[s'_1, \ldots, s'_k]}\ \mathbf{C}[-] \text{ a closed context with } k \text{ holes}$$

Furthermore, the rules Sym, Axiom and Cong for the relation $\approx_E$ can be merged into a single rule, to yield a rewriting-style deduction system. Indeed, by an induction on the depth of proof trees, one can easily show that the relation $\approx_E$ on $T_\Sigma X$ given by the rules in (7.5) coincides with the relation $\approx_E^{\mathsf{R}}$ on $T_\Sigma X$ generated by the following rewriting-style rules:

$$\mathsf{Ref}\ \frac{}{s \approx_E^{\mathsf{R}} s}\ s \in T_\Sigma X \qquad \mathsf{Trans}\ \frac{s \approx_E^{\mathsf{R}} s' \quad s' \approx_E^{\mathsf{R}} s''}{s \approx_E^{\mathsf{R}} s''}$$

$$\mathsf{Rw}\ \frac{}{\mathbf{C}[t\{v \mapsto s_v\}_{v\in V}] \ \approx_E^{\mathsf{R}}\ \mathbf{C}[t'\{v \mapsto s_v\}_{v\in V}]}\ \begin{array}{l}(V \vdash t \equiv t') \in E \cup E^{\mathrm{op}},\ \{\, s_v \,\}_{v\in V} \in (T_\Sigma X)^V, \\ \mathbf{C}[-] \text{ a context with one hole} \\ \qquad \text{(possibly with variables from } X\text{)}\end{array}$$

where $E^{\mathrm{op}} = \{\, (V \vdash t \equiv t') \mid (V \vdash t' \equiv t) \in E \,\}$.

From the internal completeness of the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$, we have the following sound and complete equational reasoning by rewriting:

$$\begin{array}{rl} & \mathbb{T}\text{-}\mathbf{Alg} \models (U \vdash s \equiv s') \\ \iff & \mathbb{T}\text{-}\mathbf{Alg} \models (\langle\!\langle s \rangle\!\rangle \equiv \langle\!\langle s' \rangle\!\rangle : 1 \to T_\Sigma U) \\ \iff & \mathsf{q}_U^{\mathbb{S}} \circ \langle\!\langle s \rangle\!\rangle = \mathsf{q}_U^{\mathbb{S}} \circ \langle\!\langle s' \rangle\!\rangle : 1 \to T_{\mathbb{S}} U \\ \iff & [s]_{\approx_E} = [s']_{\approx_E} \quad \text{in } T_\Sigma U/_{\approx_E} \\ \iff & s \approx_E s' \quad \text{in } T_\Sigma U \\ \iff & s \approx_E^{\mathsf{R}} s' \quad \text{in } T_\Sigma U \end{array}$$

We conclude this example by noting that the logic derived from the TEL for algebraic theories given in (7.1) is easily shown to be complete, as a proof of $s \approx_E s'$ for $s, s' \in T_\Sigma U$ constructed by the rules given in (7.5), by an easy induction, can be turned into a proof of $U \vdash s \equiv s'$ in the logic given in (7.1).

Chapter 8

Applications

We develop term equational systems and logics for multi-sorted algebraic theories (see Section 8.1) and nominal equational theories (see Section 8.2), following the methodology below.

1. Select a TES-universe $(\mathscr{C}, \mathscr{V}, *)$ and consider within it a notion of signature such that every signature Σ gives rise to a TES-syntax $\mathbf{T}_\Sigma$ whose underlying endofunctor preserves epimorphisms and colimits of ω-chains.

2. Select a class of arities A that are projective and ω-compact and a class of coarities C, and give a syntactic description of the TES-terms $C \to \mathbf{T}_\Sigma A$. This yields a syntactic notion of equational theory with an associated model theory arising from that of term equational systems.

3. Synthesize a deduction system for equational reasoning on syntactic terms with rules arising as syntactic counterparts of the rules from the term equational logic associated to the underlying term equational system. By construction, soundness is guaranteed.

4. In view of the internal completeness result, analyze the inductive construction of free algebras to synthesize a complete equational logic by rewriting. This complete logic may be used to show the completeness of the above equational logic arising from TEL.

8.1 Multi-sorted algebraic theories

We show how multi-sorted algebraic theories (see *e.g.* [Goguen and Meseguer 1985, Climent Vidal and Soliveres Tur 2005]) arise as TESs, and derive deductive and rewriting equational logics for them respectively from the term equational logic and the construction of free algebras.

8.1.1 Multi-sorted algebraic theories

We briefly review the multi-sorted version of algebraic theories.

An $\mathbf{S}$-*sorted signature* Σ, for $\mathbf{S}$ a set of sorts, is specified by a family of sets of operators $\{\, \Sigma(\mathsf{s}_1, .., \mathsf{s}_k; \mathsf{t}) \,\}_{(\mathsf{s}_1,..,\mathsf{s}_k)\in\mathbf{S}^*, \mathsf{t}\in\mathbf{S}}$, where the elements of $\Sigma(\mathsf{s}_1 \ldots \mathsf{s}_k; \mathsf{t})$ stand for operators of arity $\mathsf{s}_1, \ldots, \mathsf{s}_k \to \mathsf{t}$.

For the family $V = \{\, V_\mathsf{s} \,\}_{\mathsf{s}\in\mathbf{S}}$ of sets consisting of variables of each sort, the family of sets $\{\, \mathsf{T}_\Sigma(V; \mathsf{t}) \,\}_{\mathsf{t}\in\mathbf{S}}$ of *terms* of each sort with variables in V is inductively built up by the following grammar:

$$\begin{array}{rcll} t \in \mathsf{T}_\Sigma(V;\mathsf{t}) & ::= & v & \text{with } v \in V_\mathsf{t} \\ & \mid & \mathsf{o}(t_1, .., t_k) & \text{with } \mathsf{o} \in \Sigma(\mathsf{s}_1 .. \mathsf{s}_k; \mathsf{t}), t_1 \in \mathsf{T}_\Sigma(V;\mathsf{s}_1), .., t_k \in \mathsf{T}_\Sigma(V;\mathsf{s}_k)\,. \end{array} \tag{8.1}$$

An *equation* of sort $\mathsf{t} \in \mathbf{S}$ on an $\mathbf{S}$-indexed family of sets V for an $\mathbf{S}$-sorted signature Σ, written $\Sigma \rhd V \vdash l \equiv r : \mathsf{t}$, is simply defined as a pair of terms $l, r \in \mathsf{T}_\Sigma(V; \mathsf{t})$.

We say that an $\mathbf{S}$-indexed family of sets $\{\, V_\mathsf{s} \,\}_{\mathsf{s}\in\mathbf{S}}$ is *finitely presentable* when the disjoint union $\biguplus_{\mathsf{s}\in\mathbf{S}} V_\mathsf{s}$ of each component is a finite set. An $\mathbf{S}$-*sorted algebraic theory* $\mathbb{T} = (\Sigma, E)$ is given by an $\mathbf{S}$-sorted signature Σ together with a set E of equations on finitely presentable $\mathbf{S}$-indexed families of sets.

An *algebra for an* $\mathbf{S}$-*sorted signature* Σ is a pair $(X, [\![-]\!])$ consisting of an $\mathbf{S}$-indexed family of carrier sets $X = \{\, X_\mathsf{s} \,\}_{\mathsf{s}\in\mathbf{S}}$ and interpretation functions $[\![\mathsf{o}]\!] : X_{\mathsf{s}_1} \times \ldots \times X_{\mathsf{s}_k} \to X_\mathsf{t}$ for each operator $\mathsf{o} \in \Sigma(\mathsf{s}_1, .., \mathsf{s}_k; \mathsf{t})$. A *homomorphism* of algebras for Σ from $(X, [\![-]\!])$ to $(X', [\![-]\!]')$ is an $\mathbf{S}$-indexed family of functions $h = \{\, h_\mathsf{s} : X_\mathsf{s} \to X'_\mathsf{s} \,\}_{\mathsf{s}\in\mathbf{S}}$ between their carrier sets that commutes with the interpretation of each operator; that is, such that $h_\mathsf{t}([\![\mathsf{o}]\!](x_1, \ldots, x_k)) = [\![\mathsf{o}]\!]'(h_{\mathsf{s}_1}(x_1), \ldots, h_{\mathsf{s}_k}(x_k))$ for each operator $\mathsf{o} \in \Sigma(\mathsf{s}_1, .., \mathsf{s}_k; \mathsf{t})$ and all $x_1 \in X_{\mathsf{s}_1}, \ldots, x_k \in X_{\mathsf{s}_k}$. Algebras and homomorphisms form the *category* Σ-$\mathbf{Alg}$ *of algebras for the signature* Σ.

By structural induction, such an algebra $(X, [\![-]\!])$ induces interpretations

$$[\![t]\!] : \textstyle\prod_{\mathsf{s}\in\mathbf{S}} {X_\mathsf{s}}^{V_\mathsf{s}} \to X_\mathsf{t}$$

of terms $t \in \mathsf{T}_\Sigma(V; \mathsf{t})$ as follows:

$$[\![t]\!] = \begin{cases} \prod_{\mathsf{s}\in\mathbf{S}} {X_\mathsf{s}}^{V_\mathsf{s}} \xrightarrow{\pi_\mathsf{t}} {X_\mathsf{t}}^{V_\mathsf{t}} \xrightarrow{\pi_v} X_\mathsf{t} & \text{for } t = v \in V_\mathsf{t} \\ \prod_{\mathsf{s}\in\mathbf{S}} {X_\mathsf{s}}^{V_\mathsf{s}} \xrightarrow{\langle [\![t_1]\!], .., [\![t_k]\!] \rangle} X_{\mathsf{s}_1} \times \ldots \times X_{\mathsf{s}_k} \xrightarrow{[\![\mathsf{o}]\!]} X_\mathsf{t} & \text{for } t = \mathsf{o}(t_1, .., t_k), \mathsf{o} \in \Sigma(\mathsf{s}_1 .. \mathsf{s}_k; \mathsf{t}) \end{cases} \tag{8.2}$$

A Σ-algebra $(X, [\![-]\!])$ is said to *satisfy* an equation $\Sigma \rhd V \vdash l \equiv r : \mathsf{t}$ whenever the interpretations of the terms l and r coincide, *i.e.*, $[\![l]\!]\vec{x} = [\![r]\!]\vec{x}$ for all $\vec{x} \in \prod_{\mathsf{s}\in\mathbf{S}} {X_\mathsf{s}}^{V_\mathsf{s}}$. An *algebra for an* $\mathbf{S}$-*sorted theory* $\mathbb{T} = (\Sigma, E)$ is an algebra for the signature Σ that satisfies every equation in E. The *category* $\mathbb{T}$-$\mathbf{Alg}$ *of algebras for the theory* $\mathbb{T}$ is the full subcategory of Σ-$\mathbf{Alg}$ consisting of the algebras for $\mathbb{T}$.

Example 8.1.1. We consider the following $\mathbf{S}_{\mathsf{BF}}$-sorted algebraic theory $\mathbb{T}_{\mathsf{BF}} = (\Sigma_{\mathsf{BF}}, E_{\mathsf{BF}})$, which we take from [Goguen and Meseguer 1985]. This example will be used later to point out a subtlety in equational reasoning for multi-sorted algebraic theories.

$$
\begin{array}{rcl}
\mathbf{S}_{\mathsf{BF}} & = & \{\ \mathsf{Bool}\ ,\ \mathsf{Foo}\ \} \\
\Sigma_{\mathsf{BF}} & = & \{\ \begin{array}{rcl} \mathsf{false} & : & \mathsf{Bool}\ , \\ \mathsf{not} & : & \mathsf{Bool} \to \mathsf{Bool}\ , \\ \mathsf{and} & : & \mathsf{Bool}, \mathsf{Bool} \to \mathsf{Bool}\ , \\ \mathsf{foo} & : & \mathsf{Foo} \to \mathsf{Bool}\ \} \end{array} \\
E_{\mathsf{BF}} & = & \{\ \begin{array}{rcl} x : \mathsf{Bool} & \vdash & \mathsf{not}(\mathsf{not}(x)) \equiv x \ :\ \mathsf{Bool}\ , \\ x : \mathsf{Bool} & \vdash & \mathsf{and}(x, \mathsf{not}(x)) \equiv \mathsf{false}\ :\ \mathsf{Bool}\ , \\ x : \mathsf{Bool} & \vdash & \mathsf{and}(x, x) \equiv x \ :\ \mathsf{Bool}\ , \\ y : \mathsf{Foo} & \vdash & \mathsf{foo}(y) \equiv \mathsf{not}(\mathsf{foo}(y))\ :\ \mathsf{Bool}\ \} \end{array}
\end{array}
$$

An algebra $(X, [\![-]\!])$ for the theory $\mathbb{T}_{\mathsf{BF}}$ consists of a pair of sets $X = (X_{\mathsf{Bool}}, X_{\mathsf{Foo}}) \in \mathbf{Set}^{\mathbf{S}_{\mathsf{BF}}}$ together with interpretation functions

$$
\begin{array}{rcl}
[\![\mathsf{false}]\!] & : & 1 \to X_{\mathsf{Bool}}\ , \\
[\![\mathsf{not}]\!] & : & X_{\mathsf{Bool}} \to X_{\mathsf{Bool}}\ , \\
[\![\mathsf{and}]\!] & : & X_{\mathsf{Bool}} \times X_{\mathsf{Bool}} \to X_{\mathsf{Bool}}\ , \\
[\![\mathsf{foo}]\!] & : & X_{\mathsf{Foo}} \to X_{\mathsf{Bool}}
\end{array}
$$

satisfying the equations in E; that is, such that

$$
\begin{array}{lrcll}
\text{for all } x \in X_{\mathsf{Bool}} & [\![\mathsf{not}]\!]([\![\mathsf{not}]\!](x)) & = & x & \text{in } X_{\mathsf{Bool}}\ , \\
\text{for all } x \in X_{\mathsf{Bool}} & [\![\mathsf{and}]\!](x, [\![\mathsf{not}]\!](x)) & = & [\![\mathsf{false}]\!]() & \text{in } X_{\mathsf{Bool}}\ , \\
\text{for all } x \in X_{\mathsf{Bool}} & [\![\mathsf{and}]\!](x, x) & = & x & \text{in } X_{\mathsf{Bool}}\ , \\
\text{for all } y \in X_{\mathsf{Foo}} & [\![\mathsf{foo}(y)]\!] & = & [\![\mathsf{not}]\!]([\![\mathsf{foo}(y)]\!]) & \text{in } X_{\mathsf{Bool}}\ .
\end{array}
$$

8.1.2 Representation as term equational systems

We encode multi-sorted algebraic theories into TESs preserving their models. To this end, let $\mathbb{T} = (\Sigma, E)$ be an $\mathbf{S}$-sorted algebraic theory for $\mathbf{S}$ a set of sorts.

Recalling the notion of product TES-universe from Example 6.3.2 (4), we consider as a universe for the theory $\mathbb{T}$ the product TES-universe $(\mathbf{Set}^{\mathbf{S}}, \mathbf{Set}, \boxtimes)$ with $P \boxtimes \{\, X_{\mathsf{s}}\,\}_{\mathsf{s}\in\mathbf{S}}$ given by $\{\, P \times X_{\mathsf{s}}\,\}_{\mathsf{s}\in\mathbf{S}}$.

We obtain a TES-syntax for the theory $\mathbb{T}$ as follows. The signature Σ induces the endofunctor F_Σ on $\mathbf{Set}^{\mathbf{S}}$ defined by setting

$$
(F_\Sigma X)_{\mathsf{t}} = \coprod_{\mathsf{o}\in\Sigma(\mathsf{s}_1,\ldots,\mathsf{s}_k;\mathsf{t})} X_{\mathsf{s}_1} \times \ldots \times X_{\mathsf{s}_k} \qquad \text{for } X \in \mathbf{Set}^{\mathbf{S}}, \mathsf{t} \in \mathbf{S}
$$

preserving the notion of model, as Σ-$\mathbf{Alg} \cong F_\Sigma$-$\mathbf{Alg}$. Since colimits in $\mathbf{Set}^{\mathbf{S}}$ can be calculated pointwise and finite limits commute with filtered colimits in $\mathbf{Set}$, one can easily show that the functor F_Σ preserves filtered colimits. The epicontinuity of F_Σ is trivial. Thus, by Theorem 3.3.1, it follows that the category F_Σ-**Alg** is monadic over $\mathbf{Set}^{\mathbf{S}}$. Furthermore, from the inductive construction of free F_Σ-algebras given in Theorem 3.2.6, one sees that the induced monad $\mathbf{T}_\Sigma = (T_\Sigma, \eta, \mu)$ on $\mathbf{Set}^{\mathbf{S}}$ is also given syntactically by the grammar (8.1); that is, the following holds:

$$(T_\Sigma X)_{\mathsf{t}} = \mathsf{T}_\Sigma(X;\mathsf{t}) \quad \text{for } X \in \mathbf{Set}^{\mathbf{S}}, \mathsf{t} \in \mathbf{S}\,.$$

The endofunctor F_Σ a the canonical strength $\mathsf{st} : P \boxtimes F_\Sigma(X) \to F_\Sigma(P \boxtimes X)$ of which the t-component for each sort $\mathsf{t} \in \mathbf{S}$

$$P \times \coprod_{\mathsf{o} \in \Sigma(\mathsf{s}_1,..,\mathsf{s}_k;\mathsf{t})} X_{\mathsf{s}_1} \times \ldots \times X_{\mathsf{s}_k} \quad \longrightarrow \quad \coprod_{\mathsf{o} \in \Sigma(\mathsf{s}_1,..,\mathsf{s}_k;\mathsf{t})} (P \times X_{\mathsf{s}_1}) \times \ldots \times (P \times X_{\mathsf{s}_k})$$

sends a pair $(p, \iota_{s_1,..,s_k,\mathsf{o}}(x_1,..,x_k))$ to $\iota_{s_1,..,s_k,\mathsf{o}}((p,x_1),..,(p,x_k))$. Following the parameterized induction scheme (6.1) of Proposition 6.3.4, for the monad $\mathbf{T}_\Sigma$ we have a strength $\widehat{\mathsf{st}} : P \boxtimes T_\Sigma(X) \to T_\Sigma(P \boxtimes X)$ of which the t-component for each sort $\mathsf{t} \in \mathbf{S}$ maps a pair $(p,t) \in P \times \mathsf{T}_\Sigma(X;\mathsf{t})$ to the term $t\{x \mapsto (p,x)\}_{\mathsf{s} \in \mathbf{S}, x \in X_{\mathsf{s}}} \in \mathsf{T}_\Sigma(P \boxtimes X;\mathsf{t})$ obtained by simultaneously substituting (p,x) for each variable x in the term t.

By definition, every equation $(V \vdash l \equiv r : \mathsf{t})$ for the signature Σ is given as a pair of terms $l, r \in (T_\Sigma V)_{\mathsf{t}}$, which is equivalently represented by a pair of maps $\langle\!\langle l \rangle\!\rangle, \langle\!\langle r \rangle\!\rangle$: $\langle\!\langle \mathsf{t} \rangle\!\rangle \to T_\Sigma V$ for $\langle\!\langle \mathsf{t} \rangle\!\rangle \in \mathbf{Set}^{\mathbf{S}}$ defined by setting $\langle\!\langle \mathsf{t} \rangle\!\rangle_{\mathsf{t}} = 1$ and $\langle\!\langle \mathsf{t} \rangle\!\rangle_{\mathsf{s}} = \emptyset$ for $\mathsf{s} \neq \mathsf{t}$. We thus encode the algebraic theory $\mathbb{T}$ as the TES

$$\langle\!\langle \mathbb{T} \rangle\!\rangle = (\mathbf{Set}^{\mathbf{S}}, \mathbf{Set}, \boxtimes, \mathbf{T}_\Sigma, \langle\!\langle E \rangle\!\rangle)$$

with the set of TES-equations $\langle\!\langle E \rangle\!\rangle$ given by $\{\langle\!\langle l \rangle\!\rangle \equiv \langle\!\langle r \rangle\!\rangle : \langle\!\langle \mathsf{t} \rangle\!\rangle \to T_\Sigma V \mid (V \vdash l \equiv r : \mathsf{t}) \in E\}$.

The TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$ is shown to be ω-inductive as follows. The base category $\mathbf{Set}^{\mathbf{S}}$ is cocomplete, as so is $\mathbf{Set}$. As the endofunctor F_Σ is ω-cocontinuous and epicontinuous, so is the monad $\mathbf{T}_\Sigma$ by Theorem 3.3.1. The arities of TES-equations in $\langle\!\langle E \rangle\!\rangle$ are projective and ω-compact: for every finitely presentable $\mathbf{S}$-indexed family of sets $V \in \mathbf{Set}^{\mathbf{S}}$, the functor $\mathbf{Set}^{\mathbf{S}}(V, -) \cong \prod_{\mathsf{s} \in \mathbf{S}} (-)_{\mathsf{s}}{}^{V_{\mathsf{s}}}$ from $\mathbf{Set}^{\mathbf{S}}$ to $\mathbf{Set}$ is obviously epicontinuous; and ω-cocontinuous, as colimits in $\mathbf{Set}^{\mathbf{S}}$ can be calculated pointwise and finite limits commute with filtered colimits in $\mathbf{Set}$.

We finally show that the encoding of the theory $\mathbb{T}$ into the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$ preserves their models; that is, that $\langle\!\langle \mathbb{T} \rangle\!\rangle$-**Alg** $\cong$ $\mathbb{T}$-**Alg**. By definition, a $\langle\!\langle \mathbb{T} \rangle\!\rangle$-algebra is an Eilenberg-Moore algebra $(X, s : T_\Sigma X \to X)$ for the monad $\mathbf{T}_\Sigma$ such that the following diagram commutes for every equation $V \vdash t_1 \equiv t_2 : \mathsf{t}$ in E:

$$\mathbf{Set}^{\mathbf{S}}(V,X) \boxtimes \langle\!\langle \mathsf{t} \rangle\!\rangle \underset{\mathbf{Set}^{\mathbf{S}}(V,X) \boxtimes \langle\!\langle t_2 \rangle\!\rangle}{\overset{\mathbf{Set}^{\mathbf{S}}(V,X) \boxtimes \langle\!\langle t_1 \rangle\!\rangle}{\rightrightarrows}} \mathbf{Set}^{\mathbf{S}}(V,X) \boxtimes T_\Sigma V$$

$$\xrightarrow{\widehat{\mathsf{st}}_{\mathbf{Set}^{\mathbf{S}}(V,X),V}} T_\Sigma\big(\mathbf{Set}^{\mathbf{S}}(V,X) \boxtimes V\big) \xrightarrow{T_\Sigma(\epsilon^V_X)} T_\Sigma X \xrightarrow{\ s\ } X\,.$$

It can be easily shown that the commutativity of the above diagram amounts to the following:

$$\text{for all maps } v : V \to X \text{ in } \mathbf{Set}^{\mathbf{S}}, \quad \langle\!\langle \mathsf{t} \rangle\!\rangle \underset{\langle\!\langle t_2 \rangle\!\rangle}{\overset{\langle\!\langle t_1 \rangle\!\rangle}{\rightrightarrows}} T_\Sigma V \xrightarrow{T_\Sigma(v)} T_\Sigma X \xrightarrow{s} X \quad \text{commutes.}$$

Let $(X, \overline{[\![-]\!]})$ be the Eilenberg-Moore algebra for the monad $\mathbf{T}_\Sigma$ corresponding to a Σ-algebra $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma(\mathsf{s}_1,..,\mathsf{s}_k;\mathsf{t})})$ via the isomorphism $\Sigma\text{-}\mathbf{Alg} \cong \mathscr{C}^{\mathbf{T}_\Sigma}$. Then, it is easily seen that the Eilenberg-Moore algebra $(X, \overline{[\![-]\!]})$ satisfies the above condition if and only if the Σ-algebra $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma(\mathsf{s}_1,..,\mathsf{s}_k;\mathsf{t})})$ satisfies the equation $V \vdash t_1 \equiv t_2 : \mathsf{t}$. Thus, it follows that $\langle\!\langle \mathbb{T} \rangle\!\rangle$-$\mathbf{Alg}$ is isomorphic to the category $\mathbb{T}$-$\mathbf{Alg}$ of algebras for the $\mathbf{S}$-sorted algebraic theory $\mathbb{T}$.

8.1.3 Equational reasoning by deduction

For an $\mathbf{S}$-sorted algebraic theory $\mathbb{T} = (\Sigma, E)$, we derive the following sound equational logic from the term equational logic of the associated TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$:

$$\mathsf{Ref}\ \frac{}{V \vdash t \equiv t : \mathsf{t}}\ t \in (T_\Sigma V)_\mathsf{t} \qquad \mathsf{Sym}\ \frac{V \vdash t \equiv t' : \mathsf{t}}{V \vdash t' \equiv t : \mathsf{t}} \qquad \mathsf{Trans}\ \frac{V \vdash t \equiv t' : \mathsf{t} \quad V \vdash t' \equiv t'' : \mathsf{t}}{V \vdash t \equiv t'' : \mathsf{t}}$$

$$\mathsf{Axiom}\ \frac{}{V \vdash t \equiv t' : \mathsf{t}}\ (V \vdash t \equiv t' : \mathsf{t}) \in E \tag{8.3}$$

$$\mathsf{Subst}\ \frac{U \vdash t \equiv t' : \mathsf{t} \qquad \{\, V \vdash s_u \equiv s'_u : \mathsf{s} \,\}_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}}}{V \vdash t\{u \mapsto s_u\}_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}} \;\equiv\; t'\{u \mapsto s'_u\}_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}}}$$

where $t\{u \mapsto s_u\}_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}}$ denotes the term obtained by simultaneously substituting the term s_u for each variable $u \in U_\mathsf{s}$ with $\mathsf{s} \in \mathbf{S}$ in the term t. The rules Ref, Sym, Trans and Axiom directly follow from the corresponding TEL rules. The rule Subst is derived from the TEL rules Local and Subst in the following way:

$$\frac{\langle\!\langle t \rangle\!\rangle \equiv \langle\!\langle t' \rangle\!\rangle : \langle\!\langle \mathsf{t} \rangle\!\rangle \to T_\Sigma U \qquad \dfrac{\{\, \langle\!\langle s_u \rangle\!\rangle \equiv \langle\!\langle s'_u \rangle\!\rangle : \langle\!\langle \mathsf{s} \rangle\!\rangle \to T_\Sigma V \,\}_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}}}{[\, \langle\!\langle s_u \rangle\!\rangle \,]_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}} \;\equiv\; [\, \langle\!\langle s'_u \rangle\!\rangle \,]_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}} : U \to T_\Sigma V}\ (\text{by Local})}{\langle\!\langle t \rangle\!\rangle\{[\, \langle\!\langle s_u \rangle\!\rangle \,]_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}}\} \;\equiv\; \langle\!\langle t' \rangle\!\rangle\{[\, \langle\!\langle s'_u \rangle\!\rangle \,]_{\mathsf{s}\in\mathbf{S}, u\in U_\mathsf{s}}\} : \langle\!\langle \mathsf{t} \rangle\!\rangle \to T_\Sigma V}\ (\text{by Subst})$$

8.1.4 Equational reasoning by rewriting

We synthesize a complete equational logic by rewriting for an $\mathbf{S}$-sorted algebraic theory $\mathbb{T} = (\Sigma, E)$.

As the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle = (\mathbf{Set}^{\mathbf{S}}, \mathbf{Set}, \boxtimes, \mathbf{T}_\Sigma, \langle\!\langle E \rangle\!\rangle)$ is ω-inductive and the monad $\mathbf{T}_\Sigma$ arises from free algebras for the endofunctor F_Σ, we consider the construction (7.4) for the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$. As colimits in $\mathbf{Set}^{\mathbf{S}}$ are calculated pointwise, it follows that for a family of sets $X \in \mathbf{Set}^{\mathbf{S}}$, the family of sets $(T_\Sigma X)_1 \in \mathbf{Set}^{\mathbf{S}}$ is given by the family of quotient sets $\{\, (T_\Sigma X)_\mathsf{t}/_{\approx_{1,\mathsf{t}}} \,\}_{\mathsf{t}\in\mathbf{S}}$ of $T_\Sigma X$ under the family of equivalence relations $\{\, \approx_{1,\mathsf{t}}$ on $(T_\Sigma X)_\mathsf{t} \,\}_{\mathsf{t}\in\mathbf{S}}$ generated by the following rule:

$$\frac{}{t\{v \mapsto s_v\}_{\mathsf{s}\in\mathbf{S}, v\in V_\mathsf{s}} \;\approx_{1,\mathsf{t}}\; t'\{v \mapsto s_v\}_{\mathsf{s}\in\mathbf{S}, v\in V_\mathsf{s}}}\ \begin{array}{l}(V \vdash t \equiv t' : \mathsf{t}) \in E, \\ \big(\{\, s_v \in (T_\Sigma X)_\mathsf{s} \,\}_{\mathsf{s}\in\mathbf{S}, v\in V_\mathsf{s}}\big) \in \mathbf{Set}^{\mathbf{S}}(V, T_\Sigma X)\end{array}$$

The t-component of the map $q_0 : T_\Sigma X \twoheadrightarrow (T_\Sigma X)_1$ for each $\mathsf{t} \in \mathbf{S}$ sends a term $t \in (T_\Sigma X)_\mathsf{t}$ to its equivalence class $[t]_{\approx_{1,\mathsf{t}}} \in (T_\Sigma X)_\mathsf{t}/_{\approx_{1,\mathsf{t}}}$.

By inductively analyzing the construction of the maps q_n for $n \geq 1$, we have that the family of sets $(T_\Sigma X)_n$ for $n \geq 2$ are given by the family of quotient sets $\{\, (T_\Sigma X)_\mathsf{t}/_{\approx_{n,\mathsf{t}}} \,\}_{\mathsf{t}\in\mathbf{S}}$ of $T_\Sigma X$ under the family of equivalence relations $\{\, \approx_{n,\mathsf{t}}$ on $(T_\Sigma X)_\mathsf{t} \,\}_{\mathsf{t}\in\mathbf{S}}$ inductively generated by the following rules:

$$\frac{s \approx_{n-1,\mathsf{t}} s'}{s \approx_{n,\mathsf{t}} s'} \qquad\qquad \frac{s_1 \approx_{n-1,\mathsf{s}_1} s_1',\ \ldots,\ s_k \approx_{n-1,\mathsf{s}_k} s_k'}{\mathsf{o}(s_1,\ldots,s_k) \approx_{n,\mathsf{t}} \mathsf{o}(s_1',\ldots,s_k')}\ \mathsf{o} \in \Sigma(\mathsf{s}_1..\mathsf{s}_k;\mathsf{t})$$

The t-component of the map $q_n : (T_\Sigma X)_n \twoheadrightarrow (T_\Sigma X)_{n+1}$ for each $\mathsf{t} \in \mathbf{S}$ and $n \geq 1$ sends $[t]_{\approx_{n,\mathsf{t}}} \in (T_\Sigma X)_\mathsf{t}/_{\approx_{n,\mathsf{t}}}$ to $[t]_{\approx_{n+1,\mathsf{t}}} \in (T_\Sigma X)_\mathsf{t}/_{\approx_{n+1,\mathsf{t}}}$.

By pointwise calculating the colimit of the chain $\{\, q_n : (T_\Sigma X)_n \twoheadrightarrow (T_\Sigma X)_{n+1} \,\}_{n\geq 0}$ in $\mathbf{Set}^\mathbf{S}$, the family of sets $T_{\langle\!\langle\mathbb{T}\rangle\!\rangle} X$ is given by the family of quotient sets $\{\, (T_\Sigma X)_\mathsf{t}/_{\approx_{E,\mathsf{t}}} \,\}_{\mathsf{t}\in\mathbf{S}}$ of $T_\Sigma X$ under the family of relations $\{\, \approx_{E,\mathsf{t}}$ on $(T_\Sigma X)_\mathsf{t} \,\}_{\mathsf{t}\in\mathbf{S}}$ generated by the following rules:

$$\mathsf{Ref}\ \frac{}{t \approx_{E,\mathsf{t}} t}\ t \in (T_\Sigma X)_\mathsf{t} \qquad \mathsf{Sym}\ \frac{t \approx_{E,\mathsf{t}} t'}{t' \approx_{E,\mathsf{t}} t} \qquad \mathsf{Trans}\ \frac{t \approx_{E,\mathsf{t}} t' \quad t' \approx_{E,\mathsf{t}} t''}{t \approx_{E,\mathsf{t}} t''}$$
$$\mathsf{Axiom}\ \frac{}{t\{v \mapsto s_v\}_{\mathsf{s}\in\mathbf{S},v\in V_\mathsf{s}} \approx_{E,\mathsf{t}} t'\{v \mapsto s_v\}_{\mathsf{s}\in\mathbf{S},v\in V_\mathsf{s}}}\ \begin{array}{l}(V \vdash t \equiv t' : \mathsf{t}) \in E,\\ \{\, s_v \in (T_\Sigma X)_\mathsf{s} \,\}_{\mathsf{s}\in\mathbf{S},v\in V_\mathsf{s}}\end{array} \tag{8.4}$$
$$\mathsf{Cong}\ \frac{s_1 \approx_{E,\mathsf{s}_1} s_1',\ \ldots,\ s_k \approx_{E,\mathsf{s}_k} s_k'}{\mathsf{o}(s_1,\ldots,s_k) \approx_{E,\mathsf{t}} \mathsf{o}(s_1',\ldots,s_k')}\ \mathsf{o} \in \Sigma(\mathsf{s}_1..\mathsf{s}_k;\mathsf{t})$$

The t-component of the map $\mathsf{q}^{\langle\!\langle\mathbb{T}\rangle\!\rangle}_X : T_\Sigma X \to T_{\langle\!\langle\mathbb{T}\rangle\!\rangle} X$ for each $\mathsf{t} \in \mathbf{S}$ sends a term $t \in (T_\Sigma X)_\mathsf{t}$ to its equivalence class $[t]_{\approx_{E,\mathsf{t}}} \in (T_\Sigma X)_\mathsf{t}/_{\approx_{E,\mathsf{t}}}$.

Furthermore, the rules Sym, Axiom and Cong for the relations $\{\, \approx_{E,\mathsf{t}} \,\}_{\mathsf{t}\in\mathbf{S}}$ can be merged into a single rule, yielding a rewriting-style deduction system. Indeed, by an induction on the depth of proof trees, one can easily show that the relations $\{\, \approx_{E,\mathsf{t}} \,\}_{\mathsf{t}\in\mathbf{S}}$ coincide with the relations $\{\, \approx^\mathsf{R}_{E,\mathsf{t}} \,\}_{\mathsf{t}\in\mathbf{S}}$ generated by the following rewriting-style rules:

$$\mathsf{Ref}\ \frac{}{t \approx^\mathsf{R}_{E,\mathsf{t}} t}\ t \in (T_\Sigma X)_\mathsf{t} \qquad \mathsf{Trans}\ \frac{t \approx^\mathsf{R}_{E,\mathsf{t}} t' \quad t' \approx^\mathsf{R}_{E,\mathsf{t}} t''}{t \approx^\mathsf{R}_{E,\mathsf{t}} t''}$$
$$\mathsf{Rw}\ \frac{}{\mathbf{C}[t\{v \mapsto s_v\}_{\mathsf{s}\in\mathbf{S},v\in V_\mathsf{s}}] \approx^\mathsf{R}_{E,\mathsf{t}} \mathbf{C}[t'\{v \mapsto s_v\}_{\mathsf{s}\in\mathbf{S},v\in V_\mathsf{s}}]}$$
$$\left[\begin{array}{l}(V \vdash t \equiv t' : \mathsf{t}') \in E \cup E^{\mathrm{op}},\ \{\, s_v \in (T_\Sigma X)_\mathsf{s} \,\}_{\mathsf{s}\in\mathbf{S},v\in V_\mathsf{s}}\,,\\ \mathbf{C}[-] \text{ a context of sort } \mathsf{t} \text{ with one hole of sort } \mathsf{t}'\\ \qquad\text{(possibly with variables from } X)\end{array}\right] \tag{8.5}$$

where $E^{\mathrm{op}} = \{\, (V \vdash t \equiv t' : \mathsf{t}') \mid (V \vdash t' \equiv t : \mathsf{t}') \in E \,\}$.

From the internal completeness of the TES $\langle\!\langle\mathbb{T}\rangle\!\rangle$, we have the following sound and complete equational reasoning by rewriting:

$$\begin{aligned}
& \mathbb{T}\text{-}\mathbf{Alg} \models (U \vdash s \equiv s' : \mathsf{t}) \\
\iff\ & \mathbb{T}\text{-}\mathbf{Alg} \models (\langle\!\langle s \rangle\!\rangle \equiv \langle\!\langle s' \rangle\!\rangle : \langle\!\langle \mathsf{t} \rangle\!\rangle \to T_\Sigma U) \\
\iff\ & \mathsf{q}_U^{\langle\!\langle \mathbb{T} \rangle\!\rangle} \circ \langle\!\langle s \rangle\!\rangle = \mathsf{q}_U^{\langle\!\langle \mathbb{T} \rangle\!\rangle} \circ \langle\!\langle s' \rangle\!\rangle : \langle\!\langle \mathsf{t} \rangle\!\rangle \to T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} U \\
\iff\ & [s]_{\approx_{E,\mathsf{t}}} = [s']_{\approx_{E,\mathsf{t}}} \ \text{ in } (T_\Sigma U)_\mathsf{t} / {\approx_{E,\mathsf{t}}} \\
\iff\ & s \approx_{E,\mathsf{t}} s' \quad \text{in } (T_\Sigma U)_\mathsf{t} \\
\iff\ & s \approx^{\mathsf{R}}_{E,\mathsf{t}} s' \quad \text{in } (T_\Sigma U)_\mathsf{t}
\end{aligned} \tag{8.6}$$

We note that the logic derived from the TEL for multi-sorted algebraic theories given in (8.3) is easily shown to be complete, as a proof of $s \approx_{E,\mathsf{t}} s'$ for $s, s' \in (T_\Sigma U)_\mathsf{t}$ constructed by the rules given in (8.4) can be inductively turned into a proof of $U \vdash s \equiv s' : \mathsf{t}$ in the logic given in (8.3).

Example 8.1.2 (continued)**.** Goguen and Meseguer pointed out in [Goguen and Meseguer 1985] that a naïve generalization of rewrite-style equational reasoning for single-sorted algebraic theories to multi-sorted ones might be unsound. We consider the problematic example here and see how our rewrite-style equational reasoning for multi-sorted algebraic theories, given in (8.6), fixes the naïve reasoning to make it sound and complete.

Recall the theory $\mathbb{T}_{\mathsf{BF}} = (\Sigma_{\mathsf{BF}}, E_{\mathsf{BF}})$ given in Example 8.1.1 and consider the equality judgement

$$\vdash\ \mathsf{false}\ \equiv\ \mathsf{not}(\mathsf{false})\ :\ \mathsf{Bool}\ .$$

The following is a naïve seemingly correct reasoning of this judgement for the theory $\mathbb{T}_{\mathsf{BF}}$.

$$\begin{aligned}
\mathsf{false} &\equiv \mathsf{and}(\mathsf{foo}(y), \mathsf{not}(\mathsf{foo}(y))) && \equiv \mathsf{and}(\mathsf{not}(\mathsf{foo}(y)), \mathsf{not}(\mathsf{foo}(y))) \\
&\equiv \mathsf{not}(\mathsf{foo}(y)) && \equiv \mathsf{not}(\mathsf{and}(\mathsf{foo}(y), \mathsf{foo}(y))) \\
&\equiv \mathsf{not}(\mathsf{and}(\mathsf{foo}(y), \mathsf{not}(\mathsf{foo}(y)))) && \equiv \mathsf{not}(\mathsf{false})
\end{aligned} \tag{8.7}$$

However, the above judgement is invalidated by the algebra $(X, [\![-]\!])$ of the theory $\mathbb{T}_{\mathsf{BF}}$ defined as follows:

$$\begin{aligned}
& X_{\mathsf{Bool}} = \{\, \mathsf{F}, \mathsf{T} \,\}\,, \\
& X_{\mathsf{Foo}} = \emptyset\,, \\
& [\![\mathsf{false}]\!]() = \mathsf{F}\,, \\
& [\![\mathsf{not}]\!](\mathsf{F}) = \mathsf{T}\,, \qquad [\![\mathsf{not}]\!](\mathsf{T}) = \mathsf{F}\,, \\
& [\![\mathsf{and}]\!](\mathsf{T}, \mathsf{T}) = \mathsf{T}\,, \quad [\![\mathsf{and}]\!](x, x') = \mathsf{F} \text{ for } x = \mathsf{F} \text{ or } x' = \mathsf{F}\,, \\
& [\![\mathsf{foo}]\!] \text{ is the unique map from the empty set.}
\end{aligned}$$

One can easily check that the algebra $(X, [\![-]\!])$ satisfies the axioms of the theory $\mathbb{T}_{\mathsf{BF}}$, but does not satisfy the judgement $\vdash \mathsf{false} \equiv \mathsf{not}(\mathsf{false}) : \mathsf{Bool}$, since the maps $[\![\vdash \mathsf{false} : \mathsf{Bool}]\!]$ and $[\![\vdash \mathsf{not}(\mathsf{false}) : \mathsf{Bool}]\!]$ of type $1 \to X_{\mathsf{Bool}}$ respectively send the element of the singleton set 1 to the elements F and T of the set X_{Bool}. Note however that $(X, [\![-]\!])$

satisfies the judgement $y : \mathsf{Foo} \vdash \mathsf{foo}(y) \equiv \mathsf{not}(\mathsf{foo}(y)) : \mathsf{Bool}$, as both interpretations $[\![y : \mathsf{Foo} \vdash \mathsf{foo}(y) : \mathsf{Bool}]\!], [\![y : \mathsf{Foo} \vdash \mathsf{not}(\mathsf{foo}(y)) : \mathsf{Bool}]\!] : X_{\mathsf{Foo}} \to X_{\mathsf{Bool}}$ are the unique maps from the empty set.

To avoid this kind of false reasoning by rewriting, as suggested in [Goguen and Meseguer 1985], one can develop a deductive equational logic such as the one given in (8.3). However, our rewriting-style reasoning for multi-sorted algebraic theories by means of (8.5) and (8.6) directly fixes the naïve reasoning. Indeed, from (8.6), we see that a valid reasoning of the judgement $\vdash \mathsf{false} \equiv \mathsf{not}(\mathsf{false}) : \mathsf{Bool}$ by rewriting should be carried out on the set $(T_{\Sigma_{\mathsf{BF}}}(\{\, \emptyset \,\}_{\mathsf{s} \in \mathbf{S}}))_{\mathsf{Bool}}$ (*i.e.*, the set consisting of terms of sort Bool with no variables). In this view, the naïve reasoning (8.7) of the judgement $\vdash \mathsf{false} \equiv \mathsf{not}(\mathsf{false}) : \mathsf{Bool}$ is not valid because terms with a variable y appear during the rewriting; rather it is a valid reasoning of the judgement $y : \mathsf{Foo} \vdash \mathsf{false} \equiv \mathsf{not}(\mathsf{false}) : \mathsf{Bool}$.

8.2 Synthetic nominal equational theories

Gabbay and Mathijssen [2006, 2007], on the one hand, and Clouston and Pitts [2007], on the other, have respectively introduced the essentially equivalent notions of *nominal algebra* and *nominal equational theory*, and presented sound and complete deductive equational logics for them.

In this section, having the notion of nominal equational theory in mind, we consider a class of TESs, which we call Nominal Equational Systems (NESs), based on the category **Nom** of nominal sets [Gabbay and Pitts 1999, 2001, Section 6] (which is equivalent to the Schanuel topos [Mac Lane and Moerdijk 1992, Section III.9]). The syntactic description of NESs gives rise to a concrete notion of equational theory based on nominal sets, which we call *synthetic nominal equational theory*; and its model theory is derived from that of NESs.

A sound deductive equational logic, called *Synthetic Nominal Equational Logic (SNEL)*, for synthetic nominal equational theories is derived from the TEL associated to NESs. Also, a sound and complete rewriting equational logic, called *synthetic nominal rewriting*, is extracted from the construction of free algebras due to the internal completeness result. By an easy induction, the completeness of SNEL follows from the completeness of synthetic nominal rewriting.

We conclude the section by discussing the equivalence between our synthetic nominal equational logic and the nominal equational logic of Clouston and Pitts [2007]; and by comparing our synthetic nominal rewriting and the nominal rewriting of Fernández et al. [2004].

Note that the development in this section easily extends to the multi-sorted case, based on the product universe.

8.2.1 Nominal sets

For a fixed countably infinite set A of atoms, the group $\mathfrak{S}_0(\mathsf{A})$ of finite permutations of atoms consists of the bijections on A that fix all but finitely many elements of A. A $\mathfrak{S}_0(\mathsf{A})$-*action* $X = (\,|X|, \cdot)$ consists of a set $|X|$ equipped with a function $(-)\cdot(=) : \mathfrak{S}_0(\mathsf{A}) \times |X| \to |X|$ satisfying $\mathrm{id}_{\mathsf{A}} \cdot x = x$ and $\pi' \cdot (\pi \cdot x) = (\pi'\pi) \cdot x$ for all $x \in |X|$ and $\pi, \pi' \in \mathfrak{S}_0(\mathsf{A})$. $\mathfrak{S}_0(\mathsf{A})$-actions form a category with morphisms $X \to Y$ given by *equivariant functions*; that is, functions $f : |X| \to |Y|$ such that $f(\pi \cdot x) = \pi \cdot (fx)$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$ and $x \in |X|$.

For a $\mathfrak{S}_0(\mathsf{A})$-action X, a finite subset S of A is said to *support* $x \in X$ if for all atoms $a, a' \notin S$, we have that $(a\ a') \cdot x = x$, where the *transposition* $(a\ a')$ is the bijection that swaps a and a', and fixes all other atoms. A *nominal set* is a $\mathfrak{S}_0(\mathsf{A})$-action in which every element has finite support. As an example, the set of atoms A becomes the *nominal set of atoms* $\mathbb{A}$ when equipped with the evaluation action $\pi \cdot a = \pi(a)$. A further example is the nominal set $\mathscr{P}_0(\mathbb{A})$ consisting of finite subsets of A with action $\pi \cdot S = \{\, \pi \cdot a \mid a \in S \,\}$. The category **Nom** is the full subcategory of the category of $\mathfrak{S}_0(\mathsf{A})$-actions consisting of nominal sets.

The supports of an element of a nominal set are closed under intersection, and we write $\mathsf{supp}_X(x)$, or simply $\mathsf{supp}(x)$, for the intersection of the supports of x in the nominal set X. For instance, we have that $\mathsf{supp}_{\mathbb{A}}(a) = \{\, a \,\}$ and $\mathsf{supp}_{\mathscr{P}_0(\mathbb{A})}(S) = S$. For elements x and y of two, possibly distinct, nominal sets X and Y, we write $x \# y$ whenever $\mathsf{supp}_X(x)$ and $\mathsf{supp}_Y(y)$ are disjoint. Thus, for $a \in \mathbb{A}$ and $x \in X$, $a \# x$ stands for $a \notin \mathsf{supp}_X(x)$; that is, *$a$ is fresh for x*. Note that the support function $\mathsf{supp}_X : |X| \to |\mathscr{P}_0(\mathbb{A})|$ for every nominal set X is equivariant, *i.e.*, $\mathsf{supp}_X(\pi \cdot x) = \pi \cdot \mathsf{supp}_X(x)$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$ and $x \in |X|$.

The category **Nom** is complete and cocomplete. In particular, for a possibly infinite family of nominal sets $\{\, X_i \,\}_{i \in I}$, the coproduct $\coprod_{i \in I} X_i$ is given by $|\coprod_{i \in I} X_i| = \coprod_{i \in I} |X_i|$ with action $\pi \cdot \iota_i(x) = \iota_i(\pi \cdot x)$; whilst the product $\prod_{i \in I} X_i$, for a finite set I, is given by $|\prod_{i \in I} X_i| = \prod_{i \in I} |X_i|$ with action $\pi \cdot \{x_i\}_{i \in I} = \{\pi \cdot x_i\}_{i \in I}$. As usual, we write X^n for $X \times \ldots \times X$ (n times).

Further, **Nom** carries a symmetric monoidal closed structure $(1, \#, [-, =])$. The *unit* 1 is the terminal object in **Nom** (*i.e.*, the singleton set consisting of the empty tuple) equipped with the unique action. The *separating tensor* $X \# Y$ is the nominal subset of $X \times Y$ with underlying set given by $\{\, (x, y) \in |X| \times |Y| \mid x \# y \,\}$. We write $X^{\# n}$ for $X \# \ldots \# X$ (n times). For instance, $\mathbb{A}^{\# n}$ consists of n-tuples of distinct atoms equipped with the pointwise action $\pi \cdot (a_1, .., a_n) = (\pi \cdot a_1, .., \pi \cdot a_n)$. Note that $X^{\# 0}$ is 1 for any nominal set X. Henceforth we write $\boldsymbol{a}^n$, or simply $\boldsymbol{a}$ when n is clear from the context, as a shorthand for a tuple $a_1, \ldots, a_n$ of distinct atoms, and thus $\{\boldsymbol{a}^n\}$ for the set $\{a_1, \ldots, a_n\}$. A *multi-transposition* $(\boldsymbol{a}^n\ \boldsymbol{b}^n)$ denotes a fixed bijection on A satisfying $(\boldsymbol{a}^n\ \boldsymbol{b}^n)(a_i) = b_i$ for $i = 1, \ldots, n$, and $(\boldsymbol{a}^n\ \boldsymbol{b}^n)(c) = c$ for $c \notin \{\boldsymbol{a}^n\} \cup \{\boldsymbol{b}^n\}$.

The separating tensor $\#$ is closed and the associated internal-hom functor is denoted $[-,=]$. In particular, the internal homs $[\mathbb{A}^{\# n}, X]$, for $n \in \mathbb{N}$ and $X \in \mathbf{Nom}$, give rise to a notion of multi-atom abstraction. Indeed, the nominal set $[\mathbb{A}^{\# n}, X]$ has underlying set given by the quotient set $|\mathbb{A}^{\# n} \times X|/_{\approx_\alpha}$ determined by the *α-equivalence relation* $\approx_\alpha$, which is defined as follows:

> $(\boldsymbol{a}, x) \approx_\alpha (\boldsymbol{b}, x')$ if and only if there exists a *fresh* $\boldsymbol{c} \in \mathbb{A}^{\# n}$ (*i.e.*, a tuple $\boldsymbol{c} \in \mathbb{A}^{\# n}$ satisfying the condition $\boldsymbol{c} \,\#\, \boldsymbol{a}, x, \boldsymbol{b}, x')$ such that $(\boldsymbol{a}\ \boldsymbol{c}) \cdot x = (\boldsymbol{b}\ \boldsymbol{c}) \cdot x'$.

The nominal set $[\mathbb{A}^{\# n}, X]$ has action $\pi \cdot [(\boldsymbol{a}, x)]_{\approx_\alpha} = [(\pi \cdot \boldsymbol{a}, \pi \cdot x)]_{\approx_\alpha}$ on its underlying set. We write $\langle \boldsymbol{a} \rangle\, x$ for the equivalence class $[(\boldsymbol{a}, x)]_{\approx_\alpha}$. Note that $\mathsf{supp}(\langle \boldsymbol{a} \rangle\, x)$ is $\mathsf{supp}(x) \backslash \{\boldsymbol{a}\}$.

8.2.2 Synthetic nominal equational theories

We specify a class of TESs, called *Nominal Equational Systems (NESs)*, by giving a TES-universe and a class of TES-syntax and TES-equations on it. We give a syntactic description of NESs and call the syntactic counterparts of NESs *synthetic nominal equational theories.*

Nominal equational systems. The TES-universe for NESs is $(\mathbf{Nom}, \mathbf{Nom}, \#)$ with both left and right homs given by $[-,=]$; *i.e.*, the one induced from the symmetric monoidal closed structure of $\mathbf{Nom}$, according to Example 6.3.2 (2).

A *NEL-signature* Σ [Clouston and Pitts 2007] is given by a family of nominal sets $\{\, \Sigma(n) \,\}_{n \in \mathbb{N}}$, each of which consists of *operators* of arity n. To each such signature Σ, we associate the strong endofunctor $(F_\Sigma, \mathsf{st}^\Sigma)$ on $\mathbf{Nom}$ given as follows:

$$
\begin{array}{rcccc}
F_\Sigma(X) & = & \coprod_{k \in \mathbb{N}} \Sigma(k) \times X^k \ , & & \\
\mathsf{st}^\Sigma_{X,Y} & : & F_\Sigma(X) \,\#\, Y & \to & F_\Sigma(X \,\#\, Y) \\
 & : & \big(\iota_n(\mathsf{o}, x_1, \ldots, x_n), y\big) & \mapsto & \iota_n\big(\mathsf{o}, (x_1, y), \ldots, (x_n, y)\big)
\end{array}
$$

for $X, Y \in \mathbf{Nom}$ and $n \in \mathbb{N}, \mathsf{o} \in \Sigma(n), x_1, \ldots, x_n \in X, y \in Y$. Since the category $\mathbf{Nom}$ is cocomplete and the functor F_Σ is ω-cocontinuous, free F_Σ-algebras exists by Theorem 3.2.6; and the associated monad $\mathbf{T}_\Sigma = (T_\Sigma, \eta^\Sigma, \mu^\Sigma)$ has strength $\widehat{\mathsf{st}}^\Sigma$ by Proposition 6.3.4. Moreover, free Σ-algebras are constructed as in (3.3) and thus we have the following inductive description of $T_\Sigma X$:

$$
\begin{array}{rcll}
t \in T_\Sigma X & ::= & x & (\, x \in X \,) \\
 & | & \mathsf{o}\,(t_1, \ldots, t_k) & (\, \mathsf{o} \in \Sigma(k),\ t_1, .., t_k \in T_\Sigma X \,)
\end{array}
\tag{8.8}
$$

with action given by $\pi \cdot x = \pi \cdot_X x$ and $\pi \cdot \mathsf{o}(t_1, \ldots, t_k) = (\pi \cdot \mathsf{o})(\pi \cdot t_1, \ldots, \pi \cdot t_k)$. The strength $\widehat{\mathsf{st}}^\Sigma$ is given by the parameterized induction scheme (6.1), and hence described as follows:

$$
\begin{array}{rcccc}
\widehat{\mathsf{st}}^\Sigma_{X,Y} & : & T_\Sigma(X) \,\#\, Y & \to & T_\Sigma(X \,\#\, Y) \\
 & : & (t, y) & \mapsto & t\{x \mapsto (x, y)\}_{x \in X}
\end{array}
$$

where $t\{x \mapsto (x,y)\}_{x \in X}$ denotes the term obtained by simultaneously substituting (x,y) for each variable x in the term t. A TES-syntax for a NES is given by the strong monad $\mathbf{T}_\Sigma = (T_\Sigma, \eta^\Sigma, \mu^\Sigma, \widehat{\mathsf{st}}^\Sigma)$ determined by a NEL-signature Σ.

As equations for NESs, we only consider TES-equations of coarity $\mathbb{A}^{\# n}$ and arity $\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i}$ for $n, \ell, n_1, \ldots, n_\ell \in \mathbb{N}$. In summary, a NES is a TES $(\mathbf{Nom}, \mathbf{Nom}, \#, \mathbf{T}_\Sigma, E)$ for Σ a NEL-signature and E a set of TES-equations of the form $\mathbb{A}^{\# n} \to T_\Sigma(\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i})$.

Synthetic nominal equational theories. As a syntactic description of NESs, we give the notion of synthetic nominal equational theory.

First, as a syntactic counterpart of the arities $\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i}$, we introduce the notion of variable context. A *variable context* V is given by a finite set of variables $|V|$ and a function $V : |V| \to \mathbb{N}$ assigning a *valence* to each variable in $|V|$. A variable context V determines the nominal set

$$\langle\!\langle V \rangle\!\rangle = \coprod_{x \in |V|} \mathbb{A}^{\# V(x)} \; .$$

For a variable context V with $|V| = \{ x_1, \ldots, x_\ell \}$ and $V(x_i) = n_i$, for $i = 1, .., \ell$, we may write V as $\{ x_1 : n_1, \ldots, x_\ell : n_\ell \}$. We also simply write $x(\boldsymbol{a})$ for an element $\iota_x(\boldsymbol{a})$ of $\langle\!\langle V \rangle\!\rangle$ and, when convenient, further abbreviate $x()$ as x.

From the following bijection, for $n \in \mathbb{N}$ and a variable context V,

$$\begin{aligned}
&\{ t : \mathbb{A}^{\# n} \to T_\Sigma \langle\!\langle V \rangle\!\rangle \} \\
&\quad \cong \; \{ t : 1 \to [\mathbb{A}^{\# n}, T_\Sigma \langle\!\langle V \rangle\!\rangle] \} \\
&\quad \cong \; \{ t \in [\mathbb{A}^{\# n}, T_\Sigma \langle\!\langle V \rangle\!\rangle] \mid \mathsf{supp}(t) = \emptyset \} \\
&\quad = \; \{ \langle \boldsymbol{a} \rangle\, t \in [\mathbb{A}^{\# n}, T_\Sigma \langle\!\langle V \rangle\!\rangle] \mid \mathsf{supp}(t) \subseteq \{\boldsymbol{a}\} \}
\end{aligned} \tag{8.9}$$

we see that a TES-term of arity $\langle\!\langle V \rangle\!\rangle$ and coarity $\mathbb{A}^{\# n}$ is determined by an α-equivalence class $\langle \boldsymbol{a} \rangle\, t$ such that $\mathsf{supp}(t) \subseteq \{\boldsymbol{a}\}$. Syntactically, the α-equivalence class $\langle \boldsymbol{a} \rangle\, t$ is described by the pair

$$(\boldsymbol{a}, t) \quad \text{for } \boldsymbol{a} \in \mathbb{A}^{\# n},\, t \in T_\Sigma \langle\!\langle V \rangle\!\rangle \text{ such that } \mathsf{supp}(t) \subseteq \{ \boldsymbol{a} \}$$

where we understand the tuple of distinct atoms $\boldsymbol{a}$ as binding atoms; and the condition $\mathsf{supp}(t) \subseteq \{ \boldsymbol{a} \}$ as saying that there are no free atoms in the term t. From the inductive description (8.8) of T_Σ and the syntactic abbreviations for elements of $\langle\!\langle V \rangle\!\rangle$, we see that the terms $t \in T_\Sigma \langle\!\langle V \rangle\!\rangle$ with $\mathsf{supp}(t) \subseteq \{ \boldsymbol{a} \}$ are inductively described as follows:

$$\begin{array}{rcll}
t & ::= & x(\boldsymbol{a}') & (\, x(\boldsymbol{a}') \in \langle\!\langle V \rangle\!\rangle \text{ such that } \{\boldsymbol{a}'\} \subseteq \{\boldsymbol{a}\} \,) \\
& \mid & \mathsf{o}(t_1, \ldots, t_k) & (\, \mathsf{o} \in \Sigma(k) \text{ such that } \mathsf{supp}(\mathsf{o}) \subseteq \{\boldsymbol{a}\} \,) \, .
\end{array}$$

Directly motivated from this observation, we define the notion of *synthetic nominal equational theory* as follows. A *nominal context* $[\boldsymbol{a}^n]V$ consists of an *atom context* $\boldsymbol{a}^n \in \mathbb{A}^{\# n}$, for $n \in \mathbb{N}$, and a variable context V. For a nominal signature Σ, a *nominal* *term* t in a nominal context $[\boldsymbol{a}^n]V$, denoted $[\boldsymbol{a}^n]V \vdash t$, is given by an element $t \in T_\Sigma\langle\!\langle V\rangle\!\rangle$ such that $\mathsf{supp}(t) \subseteq \{\, \boldsymbol{a}^n \,\}$; and a *nominal equation* $[\boldsymbol{a}]V \vdash t \equiv t'$ is given by a pair of nominal terms t and t' in the same nominal context $[\boldsymbol{a}^n]V$. A *nominal equational theory* $\mathbb{T} = (\Sigma, E)$ consists of a NEL-signature Σ and a set of *nominal equations*.

We note that the definition of synthetic nominal equational theory depends neither on the **Nom**-action structure $\#$ of the TES-universe $(\mathbf{Nom}, \mathbf{Nom}, \#)$, nor on strengths $\widehat{\mathsf{st}}^\Sigma$ of TES-syntaxes $\mathbf{T}_\Sigma$. As we will see in the next section, these structures only affect the model theory of synthetic nominal equational theories.

Finally, we see that synthetic nominal equational theories represent nominal equational systems. Each nominal context $[\boldsymbol{a}^n]V$ determines the coarity $\mathbb{A}^{\# n}$ and the arity $\langle\!\langle V\rangle\!\rangle$; and each nominal term $[\boldsymbol{a}^n]V \vdash t$ determines the TES-term

$$\langle\!\langle [\boldsymbol{a}^n]V \vdash t\rangle\!\rangle : \mathbb{A}^{\# n} \to T_\Sigma\langle\!\langle V\rangle\!\rangle$$

corresponding to the element $\langle \boldsymbol{a}^n\rangle\, t \in [\mathbb{A}^{\# n}, T_\Sigma\langle\!\langle V\rangle\!\rangle]$ via the bijection (8.9). Indeed, the equivariant function $\langle\!\langle [\boldsymbol{a}^n]V \vdash t\rangle\!\rangle$ maps $\boldsymbol{b} \in \mathbb{A}^{\# n}$ to $(\boldsymbol{a}\ \boldsymbol{b})\cdot t \in T_\Sigma\langle\!\langle V\rangle\!\rangle$. Thus, each nominal equational theory $(\mathbb{T}, E)$ determines the NES

$$\langle\!\langle \mathbb{T}\rangle\!\rangle = (\mathbf{Nom}, \mathbf{Nom}, \#, \mathbf{T}_\Sigma, \langle\!\langle E\rangle\!\rangle)$$

with the set of TES-equations $\langle\!\langle E\rangle\!\rangle$ given by $\{\, \langle\!\langle [\boldsymbol{a}]V \vdash t\rangle\!\rangle \equiv \langle\!\langle [\boldsymbol{a}]V \vdash t'\rangle\!\rangle \,\}_{(\,[\boldsymbol{a}]V \vdash t \equiv t'\,) \in E}$.

Remark 8.2.1. To have a bijection between nominal equational systems and synthetic nominal equational theories, one has to take α-equivalence classes of nominal terms for the α-equivalence relation $\approx_\alpha$ generated by the rule

$$\big([\boldsymbol{a}^n]V \vdash t\big) \ \approx_\alpha\ \big([\boldsymbol{b}^n]V \vdash (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t\big)\,.$$

However, instead of imposing the equivalence on syntactic terms, we take this into account when we reason about them by introducing the following rule:

$$\frac{[\boldsymbol{a}^n]V \vdash t \equiv t'}{[\boldsymbol{b}^n]V \vdash (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t \equiv (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t'}$$

Example 8.2.2. (*cf.* [Gabbay and Mathijssen 2007, Clouston and Pitts 2007]) The NEL-signature Σ_λ for the untyped λ-calculus is given by the nominal sets of operators

$$\begin{aligned} \Sigma_\lambda(0) &= \{\, \boldsymbol{V}_a \mid a \in \mathbb{A}\,\}\,, \\ \Sigma_\lambda(1) &= \{\, \boldsymbol{L}_a \mid a \in \mathbb{A}\,\}\,, \\ \Sigma_\lambda(2) &= \{\, \boldsymbol{A}\,\} \end{aligned}$$

with action

$$\pi\cdot\boldsymbol{V}_a = \boldsymbol{V}_{\pi(a)}\ ,\quad \pi\cdot\boldsymbol{L}_a = \boldsymbol{L}_{\pi(a)}\ ,\quad \pi\cdot\boldsymbol{A} = \boldsymbol{A}\ .$$

The nominal equational theory $\mathbb{T}_\lambda = (\Sigma_\lambda, E_\lambda)$ for $\alpha\beta\eta$-equivalence of untyped λ-terms consists of the NEL-signature Σ_λ and the set E_λ of the following equations:

$$
\begin{array}{lrcl}
(\alpha) & [a,b]\ x:1 & \vdash & \boldsymbol{L}_a.\, x(a) \;\equiv\; \boldsymbol{L}_b.\, x(b) \\
(\beta_\kappa) & [a]\ x:0, y:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, x\,,\, y(a)\big) \;\equiv\; x \\
(\beta_{\boldsymbol{V}}) & [a]\ x:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, \boldsymbol{V}_a\,,\, x(a)\big) \;\equiv\; x(a) \\
(\beta_{\boldsymbol{L}}) & [a,b]\ x:2, y:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, \boldsymbol{L}_b.\, x(a,b)\,,\, y(a)\big) \;\equiv\; \boldsymbol{L}_b.\, \boldsymbol{A}\big(\boldsymbol{L}_a.\, x(a,b)\,,\, y(a)\big) \\
(\beta_{\boldsymbol{A}}) & [a]\ x:1, y:1, z:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, \boldsymbol{A}(x(a), y(a))\,,\, z(a)\big) \\
 & & & \;\equiv\; \boldsymbol{A}\big(\boldsymbol{A}\big(\boldsymbol{L}_a.\, x(a), z(a)\big)\,,\, \boldsymbol{A}\big(\boldsymbol{L}_a.\, y(a), z(a)\big)\big) \\
(\beta_\varepsilon) & [a,b]\ x:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, x(a), \boldsymbol{V}_b\big) \;\equiv\; x(b) \\
(\eta) & [a]\ x:0 & \vdash & \boldsymbol{L}_a.\, \boldsymbol{A}(x, \boldsymbol{V}_a) \;\equiv\; x
\end{array}
$$

where we write $\boldsymbol{L}_a.\, t$ for $\boldsymbol{L}_a(t)$.

8.2.3 Model theory

A model theory for a nominal equational theory $\mathbb{T} = (\Sigma, E)$ follows from that for the NES $\langle\!\langle \mathbb{T} \rangle\!\rangle$. This we now spell out in elementary terms.

Recall that an algebra for the associated NES $\langle\!\langle \mathbb{T} \rangle\!\rangle$ is an Eilenberg-Moore algebra $(M, s : T_\Sigma M \to M)$ for the monad $\mathbf{T}_\Sigma$ such that the following diagram, for each nominal equation $[\boldsymbol{a}^n]V \vdash t_1 \equiv t_2$ in E, commutes:

$$
\begin{array}{l}
[\langle\!\langle V \rangle\!\rangle, M] \mathbin{\#} \mathbb{A}^{\# n} \underset{\mathrm{id} \,\#\, \langle\!\langle [\boldsymbol{a}]V \vdash t_2 \rangle\!\rangle}{\overset{\mathrm{id} \,\#\, \langle\!\langle [\boldsymbol{a}]V \vdash t_1 \rangle\!\rangle}{\rightrightarrows}} [\langle\!\langle V \rangle\!\rangle, M] \mathbin{\#} T_\Sigma \langle\!\langle V \rangle\!\rangle \\
\qquad \xrightarrow{\widehat{\mathsf{st}}^\Sigma_{[\langle\!\langle V \rangle\!\rangle, M], \langle\!\langle V \rangle\!\rangle}} T_\Sigma\big([\langle\!\langle V \rangle\!\rangle, M] \mathbin{\#} \langle\!\langle V \rangle\!\rangle\big) \xrightarrow{T_\Sigma(\epsilon_M^{\langle\!\langle V \rangle\!\rangle})} T_\Sigma M \xrightarrow{s} M\,.
\end{array}
$$

From the following isomorphisms

$$
\begin{gathered}
\mathbf{Nom}^{\mathbf{T}_\Sigma} \;\cong\; F_\Sigma\text{-}\mathbf{Alg}\;, \\
[\langle\!\langle V \rangle\!\rangle, M] \;=\; \big[\textstyle\coprod_{x \in |V|} \mathbb{A}^{\# V(x)},\, M\big] \;\cong\; \textstyle\prod_{x \in |V|} [\mathbb{A}^{\# V(x)}, M]\;,
\end{gathered}
$$

it follows that $\langle\!\langle \mathbb{T} \rangle\!\rangle$-algebras bijectively correspond to F_Σ-algebras $(M, \mathsf{e} : F_\Sigma M \to M)$ such that the following diagram, for each nominal equation $[\boldsymbol{a}^n]V \vdash t_1 \equiv t_2$ in E, commutes:

$$
\begin{array}{l}
\big(\prod_{x \in |V|} [\mathbb{A}^{\# V(x)}, M]\big) \mathbin{\#} \mathbb{A}^{\# n} \underset{\mathrm{id} \,\#\, \langle\!\langle [\boldsymbol{a}]V \vdash t_2 \rangle\!\rangle}{\overset{\mathrm{id} \,\#\, \langle\!\langle [\boldsymbol{a}]V \vdash t_1 \rangle\!\rangle}{\rightrightarrows}} \big(\prod_{x \in |V|} [\mathbb{A}^{\# V(x)}, M]\big) \mathbin{\#} T_\Sigma \langle\!\langle V \rangle\!\rangle \\
\xrightarrow{\widehat{\mathsf{st}}^\Sigma} T_\Sigma\Big(\big(\prod_{x \in |V|} [\mathbb{A}^{\# V(x)}, M]\big) \mathbin{\#} \langle\!\langle V \rangle\!\rangle\Big) \cong T_\Sigma\big([\langle\!\langle V \rangle\!\rangle, M] \mathbin{\#} \langle\!\langle V \rangle\!\rangle\big) \xrightarrow{T_\Sigma(\epsilon_M^{\langle\!\langle V \rangle\!\rangle})} T_\Sigma M \xrightarrow{\widehat{\mathsf{e}}} M
\end{array}
\tag{8.10}
$$

where $(M, \widehat{\mathsf{e}} : T_\Sigma M \to M)$ is the Eilenberg-Moore algebra for $\mathbf{T}_\Sigma$ corresponding to the F_Σ-algebra (M, e).

One can easily see that the condition (8.10) amounts to the interpretation of the functorial equation

$$\mathbf{Nom} : F_\Sigma \rhd F_{[\boldsymbol{a}^n]V} \vdash [\![\, [\boldsymbol{a}^n]V \vdash t_1 \,]\!] \equiv [\![\, [\boldsymbol{a}^n]V \vdash t_2 \,]\!]$$

where the endofunctor $F_{[\boldsymbol{a}^n]V}$ is defined by setting

$$F_{[\boldsymbol{a}^n]V}(M) \;=\; \big(\textstyle\prod_{x\in|V|}[\mathbb{A}^{\#\,V(x)}, M]\big) \# \mathbb{A}^{\#\,n}$$

and the functorial term $[\![\, [\boldsymbol{a}^n]V \vdash t \,]\!] : F_\Sigma\text{-}\mathbf{Alg} \to F_{[\boldsymbol{a}^n]V}\text{-}\mathbf{Alg}$, for each nominal term $[\boldsymbol{a}^n]V \vdash t$, is given as in (8.10). By analyzing the maps in (8.10), one sees that the functorial term $[\![\, [\boldsymbol{a}^n]V \vdash t \,]\!]$ sends an F_Σ-algebra (M, e) to the $F_{[\boldsymbol{a}^n]V}$-algebra $(M, [\![\, [\boldsymbol{a}^n]V \vdash t \,]\!]_{(M,\mathsf{e})})$ defined by setting, for $\big(\{\, \langle \boldsymbol{d}_x\rangle\, m_x \,\}_{x\in|V|},\, \boldsymbol{b}^n\big) \in F_{[\boldsymbol{a}^n]V}(M)$,

$$\begin{aligned}
[\![\, [\boldsymbol{a}]V \vdash x(\boldsymbol{a'}) \,]\!]_{(M,\mathsf{e})}\big(\{\, \langle \boldsymbol{d}_x\rangle\, m_x \,\}_{x\in|V|},\, \boldsymbol{b}\big) \;&=\; (\boldsymbol{d}_x\, \boldsymbol{b'}) \cdot m_x \quad \text{with } \boldsymbol{b'} = (\boldsymbol{a}\ \boldsymbol{b}) \cdot \boldsymbol{a'} \\
[\![\, [\boldsymbol{a}]V \vdash \mathsf{o}(t_1, .\,., t_k) \,]\!]_{(M,\mathsf{e})}\big(\{\, \langle \boldsymbol{d}_x\rangle\, m_x \,\}_{x\in|V|},\, \boldsymbol{b}\big) \;&=\; \mathsf{e}_k(\mathsf{o}', t'_1, .\,., t'_k)
\end{aligned}$$

where $\mathsf{e}_k : \Sigma(k) \times M^k \to M$ is the k-component of the structure map e, and

$$\mathsf{o}' = (\boldsymbol{a}\ \boldsymbol{b}) \cdot \mathsf{o}\ , \qquad t'_i = [\![\, [\boldsymbol{a}]V \vdash t_i \,]\!]_{(M,\mathsf{e})}\big(\{\, \langle \boldsymbol{d}_x\rangle\, m_x \,\}_{x\in|V|},\, \boldsymbol{b}\big)\,.$$

Now we define a $\mathbb{T}$-*algebra* for a synthetic nominal equational theory $\mathbb{T} = (\Sigma, E)$ as an F_Σ-algebra satisfying the functorial equation $F_\Sigma \rhd F_{[\boldsymbol{a}^n]V} \vdash [\![\, [\boldsymbol{a}^n]V \vdash t_1 \,]\!] \equiv [\![\, [\boldsymbol{a}^n]V \vdash t_2 \,]\!]$ for each nominal equation $[\boldsymbol{a}^n]V \vdash t_1 \equiv t_2$ in E. The category $\mathbb{T}$-$\mathbf{Alg}$ is the full subcategory of F_Σ-$\mathbf{Alg}$ consisting of $\mathbb{T}$-algebras. By construction, the category $\mathbb{T}$-$\mathbf{Alg}$ is isomorphic to the category $\langle\!\langle \mathbb{T} \rangle\!\rangle$-$\mathbf{Alg}$ for the associated NES $\langle\!\langle \mathbb{T} \rangle\!\rangle$.

Example 8.2.3 (continued)**.** For the nominal equational theory $\mathbb{T}_\lambda = (\Sigma_\lambda, E_\lambda)$ of Example 8.2.2, a $\mathbb{T}_\lambda$-algebra has a carrier $M \in \mathbf{Nom}$ with structure maps

$$\begin{aligned}
[\![\boldsymbol{V}]\!] \;&:\; \mathbb{A} \to M\,, \\
[\![\boldsymbol{L}]\!] \;&:\; \mathbb{A} \times M \to M\,, \\
[\![\boldsymbol{A}]\!] \;&:\; M^2 \to M
\end{aligned}$$

satisfying the equations of the theory. For instance, according to the equation (α), we have that

$$[\![\boldsymbol{L}]\!]\big(a, (c\ a) \cdot m\big) = [\![\boldsymbol{L}]\!]\big(b, (c\ b) \cdot m\big) \qquad \text{for all } (\langle c\rangle\, m, (a, b)) \in [\mathbb{A}, M] \# \mathbb{A}^{\#\,2}$$

and, according to the equation (η), we have that

$$[\![\boldsymbol{L}]\!]\big(a, [\![\boldsymbol{A}]\!](m, [\![\boldsymbol{V}]\!](a))\big) = m \qquad \text{for all } (m, a) \in M \# \mathbb{A}\,.$$

By examining the construction (3.4) of the free $\mathbb{T}_\lambda$-algebra over the initial F_{Σ_λ}-algebra $T_{\Sigma_\lambda}(0)$ with the syntactic structure map, one can see that the initial $\mathbb{T}_\lambda$-algebra has as carrier the nominal set of $\alpha\beta\eta$-equivalence classes of λ-terms with the appropriate $\mathfrak{S}_0(\mathsf{A})$-action.

$$\mathsf{Eqvar}\ \frac{[\boldsymbol{a}^n]V \vdash t \equiv t'}{[\boldsymbol{b}^n]V \vdash (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t \equiv (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t'}$$

$$\mathsf{Ref}\ \frac{}{[\boldsymbol{a}^n]V \vdash t \equiv t}\ [\boldsymbol{a}^n]V \vdash t \quad \mathsf{Sym}\ \frac{[\boldsymbol{a}^n]V \vdash t \equiv t'}{[\boldsymbol{a}^n]V \vdash t' \equiv t} \quad \mathsf{Trans}\ \frac{[\boldsymbol{a}^n]V \vdash t \equiv t' \quad [\boldsymbol{a}^n]V \vdash t' \equiv t''}{[\boldsymbol{a}^n]V \vdash t \equiv t''}$$

$$\mathsf{Axiom}\ \frac{}{[\boldsymbol{a}^n]V \vdash t \equiv t'}\ \big([\boldsymbol{a}^n]V \vdash t \equiv t'\big) \in E \qquad \mathsf{Elim}\ \frac{[\boldsymbol{a}^n, \boldsymbol{b}^m]V \vdash t \equiv t'}{[\boldsymbol{a}^n]V \vdash t \equiv t'}\ \big(\boldsymbol{b} \mathrel{\#} \boldsymbol{a}, t, t'\big)$$

$$\mathsf{Intro}\ \frac{[\boldsymbol{a}^n]V \vdash t \equiv t'}{[\boldsymbol{a}^n, \boldsymbol{b}^m]V^{\langle \boldsymbol{b}\rangle}\ \vdash\ t\{x(\boldsymbol{c}_x) \mapsto x(\boldsymbol{c}_x, \boldsymbol{b})\}_{x\in|V|}\ \equiv\ t'\{x(\boldsymbol{c}_x) \mapsto x(\boldsymbol{c}_x, \boldsymbol{b})\}_{x\in|V|}}\ (\boldsymbol{b} \mathrel{\#} \boldsymbol{a})$$

where $|V^{\langle \boldsymbol{b}\rangle}|\ =\ |V|$ and $\forall_{x\in|V|}\ V^{\langle \boldsymbol{b}\rangle}(x)\ =\ V(x) + m$

$$\mathsf{Subst}\ \frac{[\boldsymbol{a}^n]U \vdash t \equiv t' \qquad \{\ [{\boldsymbol{b}_x}^{U(x)}]V \vdash s_x \equiv s'_x\ \}_{x\in|U|}}{[\boldsymbol{a}^n]V\ \vdash\ t\{x(\boldsymbol{b}_x) \mapsto s_x\}_{x\in|U|}\ \equiv\ t'\{x(\boldsymbol{b}_x) \mapsto s_x\}_{x\in|U|}}$$

Figure 8.1: Rules of SNEL for $\mathbb{T} = (\Sigma, E)$.

8.2.4 Equational reasoning by deduction

For a synthetic nominal equational theory $\mathbb{T} = (\Sigma, E)$, from the term equational logic for the NES $\langle\!\langle \mathbb{T} \rangle\!\rangle$, we obtain a sound logic for the theory $\mathbb{T}$, which we call *Synthetic Nominal Equational Logic (SNEL)*. The rules of SNEL are described in Figure 8.1.

The *substitution* operation used in the rules Intro and Subst of SNEL maps nominal terms

$$t \in T_\Sigma\langle\!\langle U \rangle\!\rangle, \quad \{\ \langle \boldsymbol{c}_x \rangle\, m_x \in [\mathbb{A}^{\#\,U(x)}, T_\Sigma(M)]\ \}_{x\in|U|}$$

for a variable context U and a nominal set M, to the nominal term

$$t\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|}\ \in\ T_\Sigma(M)$$

defined by structural induction on t as follows:

$$\begin{aligned} x(\boldsymbol{a})\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|} \quad &= \quad (\boldsymbol{c}_x\ \boldsymbol{a}) \cdot m_x \\ \mathsf{o}(t_1, .., t_k)\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|} \quad &= \quad \mathsf{o}\big(t_1\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|}, \ldots, t_k\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|}\big)\,. \end{aligned}$$

We now see how each rule of SNEL is induced from the rules of TEL.

- The SNEL rule Eqvar follows from the consideration in Remark 8.2.1.
- The SNEL rules Ref, Sym, Trans and Axiom directly follows from the corresponding TEL rules.
- The SNEL rule Elim arises from the TEL rule Local with respect to the epimorphic projection map $\mathbb{A}^{\#(n+m)} \twoheadrightarrow \mathbb{A}^{\#\,n}$ sending $(\boldsymbol{a}^n, \boldsymbol{b}^m)$ to $(\boldsymbol{a}^n)$.

- The SNEL rule Intro arises from the TEL rule Ext extended with the nominal set $\mathbb{A}^{\# m}$. Note that the TES-term $\langle\!\langle [\boldsymbol{a}^n, \boldsymbol{b}^m] V^{\langle \boldsymbol{b} \rangle} \vdash t\{x(\boldsymbol{c}_x) \mapsto x(\boldsymbol{c}_x, \boldsymbol{b})\}_{x \in |V|} \rangle\!\rangle$ amounts to the composite

$$\mathbb{A}^{\#(n+m)} \cong \mathbb{A}^{\# m} \# \mathbb{A}^{\# n} \xrightarrow{\langle \mathbb{A}^{\# m} \rangle \langle\!\langle [\boldsymbol{a}] V \vdash t \rangle\!\rangle} T_\Sigma\big(\mathbb{A}^{\# m} \# \langle\!\langle V \rangle\!\rangle\big) \cong T_\Sigma\big(\textstyle\coprod_{x \in |V|} \mathbb{A}^{\#(V(x)+m)}\big) .$$

- The SNEL rule Subst arises from the TEL rule Subst together with the rule Local as follows:

$$\dfrac{\langle\!\langle [\boldsymbol{a}^n] U \vdash t \rangle\!\rangle \equiv \langle\!\langle [\boldsymbol{a}^n] U \vdash t' \rangle\!\rangle \qquad \dfrac{\{\, \langle\!\langle [\boldsymbol{b}_x{}^{U(x)}] V \vdash s_x \rangle\!\rangle \equiv \langle\!\langle [\boldsymbol{b}_x{}^{U(x)}] V \vdash s'_x \rangle\!\rangle \,\}_{x \in |U|}}{\big[\langle\!\langle [\boldsymbol{b}_x] V \vdash s_x \rangle\!\rangle\big]_{x \in |U|} \;\equiv\; \big[\langle\!\langle [\boldsymbol{b}_x] V \vdash s'_x \rangle\!\rangle\big]_{x \in |U|}} \text{(by Local)}}{\langle\!\langle [\boldsymbol{a}] U \vdash t \rangle\!\rangle \Big\{ \big[\langle\!\langle [\boldsymbol{b}_x] V \vdash s_x \rangle\!\rangle\big]_{x \in |U|} \Big\} \;\equiv\; \langle\!\langle [\boldsymbol{a}] U \vdash t' \rangle\!\rangle \Big\{ \big[\langle\!\langle [\boldsymbol{b}_x] V \vdash s'_x \rangle\!\rangle\big]_{x \in |U|} \Big\}} \text{(by Subst)}$$

  where the rule Local applies with respect to the jointly epimorphic family of maps

$$\Big\{ \iota_x : \mathbb{A}^{\# U(x)} \to \Big(\coprod_{x \in |U|} \mathbb{A}^{\# U(x)} \Big) = \langle\!\langle U \rangle\!\rangle \Big\}_{x \in |U|} .$$

  Note that the following equality holds:

$$\langle\!\langle [\boldsymbol{a}] V \vdash t\{x(\boldsymbol{b}_x) \mapsto s_x\}_{x \in |U|} \rangle\!\rangle \;=\; \langle\!\langle [\boldsymbol{a}] U \vdash t \rangle\!\rangle \Big\{ \big[\langle\!\langle [\boldsymbol{b}_x] V \vdash s_x \rangle\!\rangle\big]_{x \in |U|} \Big\} .$$

By construction, if a nominal equation $[\boldsymbol{a}^n] V \vdash t \equiv t'$ is derivable in SNEL, then the TES-equation $\langle\!\langle E \rangle\!\rangle \vdash \langle\!\langle [\boldsymbol{a}^n] V \vdash t \rangle\!\rangle \equiv \langle\!\langle [\boldsymbol{a}^n] V \vdash t' \rangle\!\rangle$ is derivable in TEL. Thus, the soundness of SNEL follows from that of TEL.

Remark 8.2.4. Since the category of sets embeds in that of nominal sets, every algebraic theory is a nominal equational theory and for them SNEL restricted to contexts with empty atom context and variables of valence zero reduces to the logic given in (7.1) for algebraic theories.

Example 8.2.5 (continued)**.** For the nominal equational theory $\mathbb{T}_\lambda$, we can prove the judgement

$$[a]\; x : 1, y : 0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_a.\, x(a), y) \;\equiv\; \boldsymbol{L}_a.\, x(a)$$

using SNEL, as follows:

$$\mathbf{A} : \dfrac{\begin{array}{rll} & [a,b]\; x : 1 \vdash \boldsymbol{L}_a.\, x(a) \;\equiv\; \boldsymbol{L}_b.\, x(b) & \text{by Axiom } (\alpha) \\ x \mapsto & [c]\; x : 1, y : 0 \vdash x(c) \;\equiv\; x(c) & \text{by Ref} \end{array}}{[a,b]\; x : 1, y : 0 \vdash \boldsymbol{L}_a.\, x(a) \;\equiv\; \boldsymbol{L}_b.\, x(b)} \text{by Subst}$$

$$\mathbf{B} : \dfrac{\begin{array}{rll} & [a,b]\; z : 2, w : 0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.z(a,b), w) \;\equiv\; \boldsymbol{A}(\boldsymbol{L}_a.z(a,b), w) & \text{by Ref} \\ z \mapsto & [a,b]\; x : 1, y : 0 \vdash \boldsymbol{L}_a.\, x(a) \;\equiv\; \boldsymbol{L}_b.\, x(b) & \text{by } \mathbf{A} \\ w \mapsto & [\,]\; x : 1, y : 0 \vdash y \;\equiv\; y & \text{by Ref} \end{array}}{[a,b]\; x : 1, y : 0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_a.\, x(a), y) \;\equiv\; \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_b.\, x(b), y)} \text{by Subst}$$

$$\mathbf{C}: \frac{[a]\ x:0, y:1 \vdash \boldsymbol{A}\big(\boldsymbol{L}_a.\, x\,,\, y(a)\big) \;\equiv\; x \quad \text{by } \mathsf{Axiom}\ (\beta_\kappa)}{[a,b]\ x:1, y:2 \vdash \boldsymbol{A}\big(\boldsymbol{L}_a.\, x(b)\,,\, y(a,b)\big) \;\equiv\; x(b)} \text{ by } \mathsf{Intro}$$

$$\mathbf{D}: \frac{\begin{array}{lll} & [a,b]\ x:1, y:2 \vdash \boldsymbol{A}\big(\boldsymbol{L}_a.\, x(b)\,,\, y(a,b)\big) \;\equiv\; x(b) & \text{by } \mathbf{C} \\ x \mapsto & [b]\ x:1, y:0 \vdash \boldsymbol{L}_b.\, x(b) \;\equiv\; \boldsymbol{L}_b.\, x(b) & \text{by } \mathsf{Ref} \\ y \mapsto & [a,b]\ x:1, y:0 \vdash y \;\equiv\; y & \text{by } \mathsf{Ref} \end{array}}{[a,b]\ x:1, y:0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_b.\, x(b), y) \;\equiv\; \boldsymbol{L}_b.\, x(b)} \text{ by } \mathsf{Subst}$$

$$\frac{[a,b]\ x:1, y:0 \vdash\ \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_a.\, x(a), y) \equiv \boldsymbol{L}_a.\, x(a) \ \text{ by } \mathsf{Trans}(\mathsf{Trans}(\mathbf{B},\mathbf{D}),\mathsf{Sym}(\mathbf{A}))}{[a]\ x:1, y:0 \vdash\ \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_a.\, x(a), y) \equiv \boldsymbol{L}_a.\, x(a)} \text{ by } \mathsf{Elim}$$

8.2.5 Equational reasoning by rewriting

We obtain a sound and complete rewriting-style deduction system for nominal equational theories, which we call *synthetic nominal rewriting.*

From the facts that finite limits commute with filtered colimits in **Nom** and that an equivariant function in **Nom** is epimorphic if and only if its underlying function in **Set** is epimorphic, one can easily show that the endofunctor F_Σ for every NEL-signature Σ is ω-cocontinuous and epicontinuous, and the nominal set $\langle\!\langle V \rangle\!\rangle$ for every variable context V is ω-compact and projective. Thus, for every nominal equational theory $\mathbb{T}$, the associated NES $\langle\!\langle \mathbb{T} \rangle\!\rangle$ is ω-inductive.

For a nominal equational theory $\mathbb{T} = (\Sigma, E)$, we consider the construction (7.4) for the associated NES $\langle\!\langle \mathbb{T} \rangle\!\rangle$. Since the forgetful functor $|{-}| : \mathbf{Nom} \to \mathbf{Set}$ creates colimits, we have the following explicit description of the construction. For a nominal set X, the nominal set $(T_\Sigma X)_1$ has as underlying set the quotient set $|T_\Sigma X|/_{\approx_1}$ under the equivalence relation $\approx_1$ on the set $|T_\Sigma X|$ generated by the following rule:

$$\frac{}{\big((\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t\big)\{x(c_x) \mapsto s_x\}_{x\in|U|} \approx_1 \big((\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t'\big)\{x(c_x) \mapsto s_x\}_{x\in|U|}}\ \big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \in E$$

for $\boldsymbol{b}^n \in \mathbb{A}^{\#\,n}$, $\{\, \langle \boldsymbol{c}_x \rangle\, s_x \in [\mathbb{A}^{\#\,U(x)}, T_\Sigma X]\,\}_{x\in|U|}$ such that $\forall_{x\in|U|}\ \boldsymbol{b}^n \# \langle \boldsymbol{c}_x \rangle\, s_x$. The underlying set $|T_\Sigma X|/_{\approx_1}$ is equipped with action $\pi \cdot [t]_{\approx_1} = [\pi \cdot t]_{\approx_1}$. The equivariant function $q_0 : T_\Sigma X \twoheadrightarrow (T_\Sigma X)_1$ maps a term t to its equivalence class $[t]_{\approx_1}$.

The nominal sets $(T_\Sigma X)_n$ for $n \geq 2$ have as underlying set the quotient sets $|T_\Sigma X|/_{\approx_n}$ under the equivalence relations $\approx_n$ on the set $|T_\Sigma X|$ generated by the following rules:

$$\frac{s \approx_{n-1} s'}{s \approx_n s'} \qquad\qquad \frac{s_1 \approx_{n-1} s_1',\ \ldots,\ s_k \approx_{n-1} s_k'}{\mathsf{o}(s_1, .., s_k) \approx_n \mathsf{o}(s_1', .., s_k')}\ \big(\mathsf{o} \in \Sigma(k)\big)$$

The underlying sets $|T_\Sigma X|/_{\approx_n}$ are equipped with action $\pi \cdot [t]_{\approx_n} = [\pi \cdot t]_{\approx_n}$. The equivariant functions $q_{n-1} : (T_\Sigma X)_{n-1} \twoheadrightarrow (T_\Sigma X)_n$ map $[t]_{\approx_{n-1}}$ to $[t]_{\approx_n}$.

The nominal set $T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} X$, being the colimit of the chain $\{\, q_n : (T_\Sigma X)_n \twoheadrightarrow (T_\Sigma X)_{n+1} \,\}_{n\geq 0}$, is given by $|T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} X| = |T_\Sigma X|/_{\approx_E}$ with action $\pi \cdot [t]_{\approx_E} = [\pi \cdot t]_{\approx_E}$ for $\approx_E$ the equivalence relation on the set $|T_\Sigma X|$ given by the rules of Figure 8.2. The quotient map $\mathsf{q}_X^{\langle\!\langle \mathbb{T} \rangle\!\rangle} : T_\Sigma X \to T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} X$ sends a term t to its equivalence class $[t]_{\approx_E}$,

$$\mathsf{Ref}\,\frac{}{t \approx_E t}\; t \in |T_\Sigma X| \qquad \mathsf{Sym}\,\frac{t \approx_E t'}{t' \approx_E t} \qquad \mathsf{Trans}\,\frac{t \approx_E t' \quad t' \approx_E t''}{t \approx_E t''}$$

$$\mathsf{Axiom}\,\frac{}{\big((\boldsymbol{a}^n\,\boldsymbol{b}^n)\cdot t\big)\{x(c_x) \mapsto s_x\}_{x\in|U|} \approx_E \big((\boldsymbol{a}^n\,\boldsymbol{b}^n)\cdot t'\big)\{x(c_x) \mapsto s_x\}_{x\in|U|}}\;\big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \in E$$

for $\boldsymbol{b}^n \in \mathbb{A}^{\#\,n}$, $\{\, \langle \boldsymbol{c}_x \rangle\, s_x \in [\mathbb{A}^{\#\,U(x)}, T_\Sigma X]\,\}_{x\in|U|}$ such that $\forall_{x\in|U|}\; \boldsymbol{b}^n \,\#\, \langle \boldsymbol{c}_x\rangle\, s_x$

$$\mathsf{Cong}\,\frac{s_1 \approx_E s_1',\ \ldots,\ s_k \approx_E s_k'}{\mathsf{o}(s_1,..,s_k) \approx_E \mathsf{o}(s_1',..,s_k')}\;\big(\mathsf{o} \in \Sigma(k)\big)$$

Figure 8.2: Rules for $\approx_E$.

Furthermore, the rules Sym, Axiom and Cong for the relation $\approx_E$ can be merged into a single rule, yielding a rewriting-style deduction system. Indeed, by an induction on the depth of proof trees, one can easily show that the relation $\approx_E$ coincides with the equivalence relation $\approx_E^{\mathsf{R}}$ generated by the following rewriting-style rules:

$$\mathsf{Ref}\,\frac{}{t \approx_E^{\mathsf{R}} t}\; t \in |T_\Sigma X| \qquad \mathsf{Trans}\,\frac{t \approx_E^{\mathsf{R}} t' \quad t' \approx_E^{\mathsf{R}} t''}{t \approx_E^{\mathsf{R}} t''}$$

$$\mathsf{Rw}\,\frac{}{\mathbf{C}\big[\big((\boldsymbol{a}^n\,\boldsymbol{b}^n)\cdot t\big)\{x(c_x) \mapsto s_x\}_{x\in|U|}\big] \;\approx_E^{\mathsf{R}}\; \mathbf{C}\big[\big((\boldsymbol{a}^n\,\boldsymbol{b}^n)\cdot t'\big)\{x(c_x) \mapsto s_x\}_{x\in|U|}\big]} \tag{8.11}$$
$$\left[\begin{array}{l} \big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \in E \cup E^{\mathrm{op}}\,, \\ \boldsymbol{b}^n \in \mathbb{A}^{\#\,n}\,,\ \{\, \langle \boldsymbol{c}_x \rangle\, s_x \in [\mathbb{A}^{\#\,U(x)}, T_\Sigma X]\,\}_{x\in|U|} \text{ such that } \forall_{x\in|U|}\; \boldsymbol{b}^n \,\#\, \langle \boldsymbol{c}_x\rangle\, s_x\,, \\ \mathbf{C}[-] \text{ a context with one hole (possibly with elements from } X) \end{array}\right]$$

where $E^{\mathrm{op}} = \big\{\ \big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \mid \big([\boldsymbol{a}^n]U \vdash t' \equiv t\big) \in E\ \big\}$. Rewriting of nominal terms by the rule (8.11) is called *synthetic nominal rewriting*.

By the internal completeness of the TES $\langle\!\langle \mathbb{T} \rangle\!\rangle$, we show the soundness and completeness of synthetic nominal rewriting as follows:

$$\begin{array}{ll} & [\boldsymbol{a}^n]V \vdash s \equiv s' \text{ is satisfied by all } \mathbb{T}\text{-algebras} \\ \iff & \langle\!\langle \mathbb{T} \rangle\!\rangle\textbf{-Alg} \;\models\; \langle\!\langle [\boldsymbol{a}^n]V \vdash s \rangle\!\rangle \equiv \langle\!\langle [\boldsymbol{a}^n]V \vdash s' \rangle\!\rangle \;:\; \mathbb{A}^{\#\,n} \to T_\Sigma \langle\!\langle V \rangle\!\rangle \\ \iff & \mathsf{q}^{\langle\!\langle \mathbb{T} \rangle\!\rangle}_{\langle\!\langle V \rangle\!\rangle} \circ \langle\!\langle [\boldsymbol{a}^n]V \vdash s \rangle\!\rangle \;=\; \mathsf{q}^{\langle\!\langle \mathbb{T} \rangle\!\rangle}_{\langle\!\langle V \rangle\!\rangle} \circ \langle\!\langle [\boldsymbol{a}^n]V \vdash s' \rangle\!\rangle \;:\; \mathbb{A}^{\#\,n} \to T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} \langle\!\langle V \rangle\!\rangle \\ \iff & \mathsf{q}^{\langle\!\langle \mathbb{T} \rangle\!\rangle}_{\langle\!\langle V \rangle\!\rangle}\big(\langle\!\langle [\boldsymbol{a}^n]V \vdash s \rangle\!\rangle(\boldsymbol{a}^n)\big) \;=\; \mathsf{q}^{\langle\!\langle \mathbb{T} \rangle\!\rangle}_{\langle\!\langle V \rangle\!\rangle}\big(\langle\!\langle [\boldsymbol{a}^n]V \vdash s' \rangle\!\rangle(\boldsymbol{a}^n)\big) \quad \text{in } |T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} \langle\!\langle V \rangle\!\rangle|/_{\approx_E} \\ \iff & [s]_{\approx_E} = [s']_{\approx_E} \quad \text{in } |T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} \langle\!\langle V \rangle\!\rangle|/_{\approx_E} \\ \iff & s \approx_E s' \quad \text{in } |T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} \langle\!\langle V \rangle\!\rangle| \\ \iff & s \approx_E^{\mathsf{R}} s' \quad \text{in } |T_{\langle\!\langle \mathbb{T} \rangle\!\rangle} \langle\!\langle V \rangle\!\rangle|\,. \end{array} \tag{8.12}$$

Example 8.2.6 (continued)**.** By synthetic nominal rewriting, we prove the judgement

$$[a]\; x : 1, y : 0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_a.\, x(a), y) \;\equiv\; \boldsymbol{L}_a.\, x(a)$$

as follows (*cf.* the proof given in Example 8.2.5):

$$
\begin{array}{lll}
& \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_a.\, x(a), y) & \\
& & \text{by } (\alpha): \begin{array}{rl} & \boldsymbol{A}(\boldsymbol{L}_a.[\ \ \boldsymbol{L}_a.\, x(a) \ \ \{x(c) \mapsto x(c)\}\ \ ], y) \\ \approx^{\mathsf{R}}_{E} & \boldsymbol{A}(\boldsymbol{L}_a.[\ \ \boldsymbol{L}_b.\, x(b) \ \ \{x(c) \mapsto x(c)\}\ \ ], y) \end{array} \\
\approx^{\mathsf{R}}_{E} & \boldsymbol{A}(\boldsymbol{L}_a.\, \boldsymbol{L}_b.\, x(b), y) & \\
& & \text{by } (\beta_\kappa): \begin{array}{rl} & [\ \ \boldsymbol{A}(\boldsymbol{L}_a.\, x, y(a)) \ \ \{x \mapsto \boldsymbol{L}_b.\, x(b); y(a) \mapsto y\}\ \ ] \\ \approx^{\mathsf{R}}_{E} & [\ \ x \ \ \{x \mapsto \boldsymbol{L}_b.\, x(b); y(a) \mapsto y\}\ \ ] \end{array} \\
\approx^{\mathsf{R}}_{E} & \boldsymbol{L}_b.\, x(b) & \\
& & \text{by } (\alpha): \begin{array}{rl} & [\ \ \boldsymbol{L}_b.\, x(b) \ \ \{x(c) \mapsto x(c)\}\ \ ] \\ \approx^{\mathsf{R}}_{E} & [\ \ \boldsymbol{L}_a.\, x(a) \ \ \{x(c) \mapsto x(c)\}\ \ ] \end{array} \\
\approx^{\mathsf{R}}_{E} & \boldsymbol{L}_a.\, x(a)\ . &
\end{array}
$$

We finally remark that one can show the completeness of SNEL by turning a proof of $s \approx_E s'$, for $s, s' \in T_\Sigma\langle\!\langle V \rangle\!\rangle$ with $\mathsf{supp}(s), \mathsf{supp}(s') \subseteq \boldsymbol{a}^n$, into a proof of $[\boldsymbol{a}^n]V \vdash s \equiv s'$ in SNEL, by a simple induction.

8.2.6 Equivalence between nominal equational logic and synthetic nominal equational logic

We discuss the logical equivalence between our synthetic nominal equational logic and the nominal equational logic of Clouston and Pitts [2007].

Nominal equational logic and synthetic nominal equational logic share the same notion of signature (*i.e.*, that of *NEL-signature*). Although their equality judgements look quite different, we provide a conversion between them in such a way that the equational constraints that they impose on algebras for NEL-signatures are preserved.

Let us start by giving an example. The judgement of synthetic nominal equational logic, representing the α-equivalence of λ-terms for the NEL-signature Σ_λ (see Example 8.2.2),

$$[a, b]\ x : 1 \ \vdash\ \boldsymbol{L}_a.\, x(a) \ \equiv\ \boldsymbol{L}_b.\, x(b)$$

is turned into the following judgement of nominal equational logic

$$\{b\} \mathrel{\#} x \ \vdash\ \boldsymbol{L}_a.\, x \ \approx\ \boldsymbol{L}_b.\, (a\ b)x\ .$$

Remark 8.2.7. The work reported in [Clouston and Pitts 2007] is based on judgements of the form

$$A_1 \mathrel{\#} x_1, \ldots, A_n \mathrel{\#} x_n \ \vdash\ A \mathrel{\#} t \approx t'$$

where $A_1, \ldots, A_n$ and A are finite sets of atoms. The sets A_i state name freshness assumptions on the variables x_i and the set A imposes name freshness conditions on the terms t and t' of the equation. However, Clouston has shown that this extension, though convenient, does not add expressive power; as every freshness judgement can be equivalently encoded as an equality judgement (see also [Gabbay and Mathijssen 2007, Theorem 5.5]). For instance, the α-equivalence axiom above is the encoding of the following one

$$\{\} \mathrel{\#} x \vdash \{a\} \mathrel{\#} \boldsymbol{L}_a\, x \approx \boldsymbol{L}_a\, x\ .$$

We now give a formal definition of nominal equational theory and explicitly describe the conversion. A *nominal equational theory* $\mathbb{T} = (\Sigma, E)$ consists of a NEL-signature $\Sigma = \{\, \Sigma(n) \,\}_{n\in\mathbb{N}}$ and a set E of equality judgements of the form

$$\{\boldsymbol{a_1}^{l_1}\} \mathrel{\#} x_1, \ldots, \{\boldsymbol{a_n}^{l_n}\} \mathrel{\#} x_n \vdash t \approx t'$$

where t and t' are terms inductively defined by the following grammar:

$$\begin{array}{rcll} t & ::= & \pi x & (\, \pi \in \mathfrak{S}_0(\mathsf{A}),\ x \in \{\, x_1, \ldots, x_n \,\}\,) \\ & | & \mathsf{o}\,(t_1, \ldots, t_k) & (\, \mathsf{o} \in \Sigma(k)\,) \end{array}$$

We also simply write x for $\mathrm{id}_{\mathsf{A}}\, x$. Recall that $\boldsymbol{a_i}^{l_i}$ denotes the tuple $a_{i\,1}, a_{i\,2}, \ldots, a_{i\,l_i}$ of distinct atoms, and thus $\{\boldsymbol{a_i}^{l_i}\}$ denotes the set $\{a_{i\,1}, \ldots, a_{i\,l_i}\}$.

An equality judgement of synthetic nominal equational logic

$$[a_1, \ldots, a_l]\; x_1 : l_1, \ldots,\, x_n : l_n \vdash t \;\equiv\; t'$$

is turned into the following judgement of nominal equational logic

$$\ldots, \{a_{l_i+1}, .., a_l\} \mathrel{\#} x_i, \ldots \vdash t\{x_i(\boldsymbol{c}^{l_i}) \mapsto (\boldsymbol{a}^{l_i}\ \boldsymbol{c}^{l_i})x_i\} \;\approx\; t'\{x_i(\boldsymbol{c}^{l_i}) \mapsto (\boldsymbol{a}^{l_i}\ \boldsymbol{c}^{l_i})x_i\}$$

where the term $t\{x_i(\boldsymbol{c}^{l_i}) \mapsto (\boldsymbol{a}^{l_i}\ \boldsymbol{c}^{l_i})x_i\}$ is obtained from the term t by simultaneously replacing all occurrences of variables $x_i(c_1, .., c_{l_i})$ with $(a_1, .., a_{l_i}\ \ c_1, .., c_{l_i})x_i$ for the multi-transposition $(a_1, .., a_{l_i}\ \ c_1, .., c_{l_i}) \in \mathfrak{S}_0(\mathsf{A})$. Conversely, an equality judgement of nominal equational logic

$$\{\boldsymbol{a_1}^{l_1}\} \mathrel{\#} x_1, \ldots, \{\boldsymbol{a_n}^{l_n}\} \mathrel{\#} x_n \vdash t \approx t'$$

is turned into the following judgement of synthetic nominal equational logic

$$[\,\boldsymbol{a}^l\,]\; \ldots, x_i : l - l_i, \ldots \vdash t\{\pi x_i \mapsto x_i\big(\pi \cdot (\boldsymbol{a} - \{\boldsymbol{a_i}\})\big)\} \;\equiv\; t'\{\pi x_i \mapsto x_i\big(\pi \cdot (\boldsymbol{a} - \{\boldsymbol{a_i}\})\big)\}$$

where $\boldsymbol{a}^l = a_1, \ldots, a_l$ is a tuple of all distinct atoms (in an arbitrary order) appearing in the judgement (*i.e.*, those appearing in t, t' and $\{\boldsymbol{a_i}^{l_i}\}$ for all $i \in \{1, .., n\}$); and where the term $t\{\pi x_i \mapsto x_i\big(\pi \cdot (\boldsymbol{a} - \{\boldsymbol{a_i}\})\big)\}$ is obtained from the term t by simultaneously substituting all occurrences of variables πx_i with $x_i\big(\pi \cdot (\boldsymbol{a} - \{\boldsymbol{a_i}\})\big)$ for $(\boldsymbol{a} - \{\boldsymbol{a_i}\}) \in \mathbb{A}^{\#(l-l_i)}$ the tuple obtained by removing the atoms $a_{i\,1}, \ldots, a_{i\,l_i}$ from the tuple $(a_1, \ldots, a_l)$.

As an another example, the equation (β_ε) of Example 8.2.2

$$[a, b]\; x : 1 \vdash \boldsymbol{A}\big(\boldsymbol{L}_a.\, x(a), \boldsymbol{V}_b\big) \;\equiv\; x(b)$$

is turned into the following one

$$\{b\} \mathrel{\#\!\!\!/} x \vdash \boldsymbol{A}\big(\boldsymbol{L}_a.\, x, \boldsymbol{V}_b\big) \;\approx\; (a\;b)x \;,$$

and vice versa.

From the model theories of nominal equational logic and synthetic nominal equational logic, it follows that an algebra for a NEL-signature Σ satisfies an equality judgement of nominal equational logic (resp. of synthetic nominal equational logic) if and only if it satisfies the equality judgment of synthetic nominal equational logic (resp. of nominal equational logic) obtained by the above conversion.

8.2.7 Comparison between nominal rewriting and synthetic nominal rewriting

Nominal rewriting [Fernández et al. 2004, Fernández and Gabbay 2007] looks like a term rewriting version of nominal equational logic [Clouston and Pitts 2007]. However, we observe that nominal rewriting—seen as an equational logic—is not complete with respect to the model theory of nominal equational theories.

Rewrite judgements for nominal rewriting systems are identical to equality judgements for nominal equational theories, except that

- the symbol "$\rightarrow$" is used in place of "$\approx$", and
- more importantly, the notion of α-equivalence is built into so called *nominal signatures* as special arities of operators; in other words, α-equivalence is given as meta-level axioms rather than object-level ones.

Remark 8.2.8. It does not make any logical difference whether one imposes α-equivalence at the object-level or at the meta-level. However, in practice, assuming it at the meta-level has advantages as α-conversion can be done in a unification process.

As an example, we consider the signature Σ_λ for the untyped λ-calculus, which consists of the operator $\boldsymbol{V}$ taking an atom, the operator $\boldsymbol{L}$ taking a term with one bound atom, and the operator $\boldsymbol{A}$ taking two terms. The terms $t \in \mathsf{T}_{\Sigma_\lambda} X$ with variables in a set X for the signature Σ_λ are inductively defined by the following grammar:

$$\begin{array}{rcll} t \in \mathsf{T}_{\Sigma_\lambda} X & ::= & \pi x & (\, \pi \in \mathfrak{S}_0(\mathsf{A}),\; x \in X\,) \\ & \mid & \boldsymbol{V}(a) & (\, a \in \mathbb{A}\,) \\ & \mid & \boldsymbol{L}(\langle a \rangle\, t) & (\, a \in \mathbb{A},\; t \in \mathsf{T}_{\Sigma_\lambda} X\,) \\ & \mid & \boldsymbol{A}(t_1, t_2) & (\, t_1, t_2 \in \mathsf{T}_{\Sigma_\lambda} X\,) \end{array}$$

From the signature for the operator $\boldsymbol{L}$, we implicitly assume the following α-equivalence axiom at the meta-level:

$$\{b\} \mathrel{\#\!\!\!/} x \vdash \boldsymbol{L}(\langle a \rangle\, x) \;\approx\; \boldsymbol{L}(\langle b \rangle\, (a\;b)x)\;. \tag{8.13}$$

A rewrite judgement for the signature Σ_λ is of the form

$$\{\boldsymbol{a_1}^{l_1}\} \not\# x_1, \ldots, \{\boldsymbol{a_n}^{l_n}\} \not\# x_n \vdash t \rightarrow t'$$

for $t, t' \in \mathsf{T}_{\Sigma_\lambda}\{x_1, \ldots, x_n\}$. As in usual term rewriting, one rewrites a term s into another term s' according to a given set $\mathcal{R}$ of rewrite rules of the above form, but under some freshness assumptions $\{\boldsymbol{b_1}^{l_1}\} \not\# y_1, \ldots, \{\boldsymbol{b_m}^{l_m}\} \not\# y_m$ on variables $\{y_1, .., y_m\} \supseteq \mathsf{Var}(s) \cup \mathsf{Var}(s')$. This is denoted as follows:

$$\{\boldsymbol{b_1}^{l_1}\} \not\# y_1, \ldots, \{\boldsymbol{b_m}^{l_m}\} \not\# y_m \vdash s \rightarrow_{\mathcal{R}} s' \ .$$

Consult [Fernández et al. 2004] for details of the relation $\rightarrow_{\mathcal{R}}$. To view nominal rewriting as an equational logic, we regard the set of rewrite rules $\mathcal{R}$ as a set of equality judgements and perform bidirectional rewriting according to $\mathcal{R}$.

We explain with a counter example that the nominal rewriting is not complete with respect to the model theory of nominal equational theories. Let $\mathcal{R}$ be the following rewrite rule for the signature Σ_λ:

$$\{a\} \not\# x, \{\} \not\# y \vdash \boldsymbol{A}\big(\boldsymbol{L}\big(\langle a\rangle\, \boldsymbol{L}(\langle b\rangle\, x)\big), y\big) \ \rightarrow\ \boldsymbol{L}(\langle b\rangle\, x) \ .$$

Considering this rule as an equality judgement, we can see that the following judgement is *valid* (*i.e.*, it is satisfied by all algebras satisfying $\mathcal{R}$):

$$\{\} \not\# x, \{\} \not\# y \vdash \boldsymbol{A}\big(\boldsymbol{L}\big(\langle a\rangle\, \boldsymbol{L}(\langle a\rangle\, x)\big), y\big) \ \approx\ \boldsymbol{L}(\langle a\rangle\, x) \ . \tag{8.14}$$

In order to rewrite the term $\boldsymbol{A}\big(\boldsymbol{L}\big(\langle a\rangle\, \boldsymbol{L}(\langle a\rangle\, x)\big), y\big)$ according to the rule $\mathcal{R}$, it needs to be α-converted to the term $\boldsymbol{A}\big(\boldsymbol{L}\big(\langle a\rangle\, \boldsymbol{L}(\langle b\rangle\, (a\ b)x)\big), y\big)$ by the meta-level axiom (8.13), which requires the assumption $\{b\} \not\# x$. Thus, it is not possible to derive the above valid judgement by means of nominal rewriting. Indeed, the incompleteness of nominal rewriting is due to the fact that it essentially lacks the rule (ATM-ELIM) of nominal equational logic [Clouston and Pitts 2007, Fig. 5]. Note that the rule (ATM-ELIM) corresponds to the rule Intro of synthetic nominal equational logic (see Figure 8.1), as the former eliminates atoms from freshness conditions (*i.e.*, it makes the atoms newly available).

We show how one can derive the judgement (8.14) by synthetic nominal rewriting. First, according to the conversion given in Section 8.2.6, the judgement is turned into the following one:

$$[a]\ x : 1,\ y : 1 \vdash \boldsymbol{A}\big(\boldsymbol{L}\big(\langle a\rangle\, \boldsymbol{L}(\langle a\rangle\, x(a))\big), y(a)\big) \ \equiv\ \boldsymbol{L}(\langle a\rangle\, x(a)) \ .$$

Then, the term $\boldsymbol{A}\big(\boldsymbol{L}\big(\langle a\rangle\, \boldsymbol{L}(\langle a\rangle\, x(a))\big), y(a)\big)$ is α-converted to $\boldsymbol{A}\big(\boldsymbol{L}\big(\langle a\rangle\, \boldsymbol{L}(\langle b\rangle\, x(b))\big), y(a)\big)$ by the meta-level axiom (8.13), as $b \# x(a)$; then it rewrites to the term $\boldsymbol{L}(\langle b\rangle\, x(b))$ by the rewrite rule $\mathcal{R}$, as $a \# x(b)$; and then it is α-converted to $\boldsymbol{L}(\langle a\rangle\, x(a))$ by the meta-level axiom (8.13), as $a \# x(b)$.

We finally remark that synthetic nominal rewriting is well suited for mechanization, as one can use the nominal unification algorithm [Urban et al. 2004] so that the meta-level α-conversion is automatically performed.

8.2.8 Comparison between binding term equational logic and nominal equational logic

We discuss how one can relate the seemingly unrelated binding term equational logic of Hamana [2003] and equational logic for nominal algebras of Gabbay and Mathijssen [2007], which is essentially equivalent to nominal equational logic of [Clouston and Pitts 2007], by viewing them as TELs.

First of all, as it is done for the nominal equational logic in Section 8.2, the equational logic for nominal algebras can be shown to be logically equivalent to a TEL based on **Nom** with TES-syntax, say $\mathbf{T}_\Sigma$, induced from nominal signatures Σ. Similarly, the binding term equational logic can be shown to essentially arise as a TEL based on $\mathbf{Set}^{\mathbb{I}}$ with TES-syntax, say $\mathbf{T}'_\Sigma$, induced from nominal signatures Σ, where $\mathbb{I}$ is the category of finite sets and injections. Moreover, the monad $\mathbf{T}_\Sigma$ is the restriction of the monad $\mathbf{T}'_\Sigma$, *i.e.*, such that $J\,\mathbf{T}_\Sigma = \mathbf{T}'_\Sigma\, J$ for the embedding $J : \mathbf{Nom} \hookrightarrow \mathbf{Set}^{\mathbb{I}}$. Thus, there is a bijection between TES-terms $C \to \mathbf{T}_\Sigma A$ and TES-terms $JC \to \mathbf{T}'_\Sigma(JA)$. Indeed, arities and coarities for the TEL based on $\mathbf{Set}^{\mathbb{I}}$ are images of arities and coarities for the TEL based on **Nom** under the embedding J, and thus the two TELs can be seen to have the same syntactic equational judgements. Furthermore, the two TELs have the same inference rules except the rule Local (see Section 7.1.1), as the projection maps $\mathbb{A}^{\#(n+m)} \to \mathbb{A}^{\#\,n}$ are epimorphic in **Nom**, but their images $J\mathbb{A}^{\#(n+m)} \to J\mathbb{A}^{\#\,n}$ are not in $\mathbf{Set}^{\mathbb{I}}$.

In conclusion, the equational logics based on **Nom** and $\mathbf{Set}^{\mathbb{I}}$ have the same syntactic judgements and inference rules except that the former has one more rule stating that unused atoms can be eliminated, which corresponds to the rule Elim of SNEL (see Figure 8.1) and the rule (ATM-INTRO) of nominal equational logic (see [Clouston and Pitts 2007, Fig. 5]).

Chapter 9

Concluding discussion

We conclude by recalling the main contributions of this thesis and discussing related work and further research directions.

9.1 Contributions

In this thesis, we generalized the classical notion of (enriched) algebraic theory to achieve sufficient expressivity as needed in modern applications, by introducing the more abstract concepts of Equational System (ES) and Term Equational System (TES). As their associated theories, we developed the construction of free algebras for ESs and equational reasoning for TESs. The concept of ES is more general than that of TES, which is still more general than that of enriched algebraic theory. Thus, both of the above developments apply to TESs and (enriched) algebraic theories.

In Part I, motivated from limitations of enriched algebraic theory in coping with modern applications, we developed the concept of equational system. One of the strengths of ES is its simplicity. The concept of ES and its associated theory require only elementary category theory. More specifically, the construction of free algebras for ESs well extends the famous construction of free algebras for endofunctors (see *e.g.* [Adámek 1974, Lehmann and Smyth 1981, Smyth and Plotkin 1982, Barr and Wells 1985, Adámek and Trnková 1990]) to an equational setting at the same level of abstraction. Further this free construction captures the intuition that free algebras consist of freely constructed terms quotiented by given equations and congruence rules. Because of its simplicity, one can easily dualize the concept, leading to the notion of equational cosystem. The concept of ES is also sufficiently general to accommodate most naturally arising equational algebraic structures. In order to show its expressivity, we have given various examples of ESs including two modern applications, Σ-monoids [Fiore et al. 1999] and π-algebras [Stark 2005, 2008]. Besides the construction of free algebras, we also extensively studied monadicity and cocompleteness of categories of algebras for ESs, providing finitary and transfinitary conditions for such properties to hold.

In Part II, we pursue a general theory of equational reasoning about algebraic structures. For this purpose, we introduced the notion of term equational system with a more concrete concept of equation, which we borrowed from enriched algebraic theories [Kelly and Power 1993]. We first developed a general equational logic, called Term Equational Logic (TEL), to reason about algebras for TESs. TEL consists of four sound deduction rules Axiom, Subst, Ext, Local together with the three equivalence relation rules Ref, Sym, Trans. Although TEL gives a complete equational logic for all our concrete examples of TESs, we do not have a general completeness result for TEL. As a further step towards completeness, we have shown internal completeness. The internal completeness result together with the inductive construction of free algebras provides an abstract process to check the validity of a given equality judgement. For concrete instances of TES, one might extract a complete rewriting-style equational logic from the abstract process. Finally, to exemplify this scenario, we exhibited two applications: multi-sorted algebraic theories and nominal equational theories of Clouston and Pitts [2007].

9.2 Related work

We have learnt during the course of this work that variations on the concept of equational system, and its dual of equational cosystem had already been considered in the literature. For instance, Fokkinga [1996] introduces the more general concept of law, but only studies initial algebras for the laws that are special cases of our concept of functorial equations. Cîrstea [2000] introduces the concept of coequation between abstract cosignatures, which is equivalent to our notion of equational cosystem, and studies final coalgebras for them. Ghani et al. [2003] introduce the concept of functorial coequational presentations, which is equivalent to our notion of equational cosystem on a locally presentable base category with an accessible functorial signature and an accessible functorial context, and study cofree constructions for them.

Our theory of equational (co)systems is more general and comprehensive than that of [Fokkinga 1996] and [Cîrstea 2000], and can be related to that of [Ghani et al. 2003] as follows. The proof of the dual of Corollary 4.1.13 (3) together with the construction of cofree coalgebras for endofunctors by terminal sequences of Worrell [1999], gives a construction of cofree coalgebras for equational cosystems on a locally presentable base category with an accessible functorial signature that preserves monomorphisms. This is a variation of a main result of the theory developed by Ghani et al. [2003] (see *e.g.* their Lemmas 5.8 and 5.14); which is proved there by means of the theory of accessible categories without assuming the preservation of monomorphisms but assuming that arities of equations are accessible endofunctors.

9.3 Further research

It is of interest to investigate the characterization of algebras for ESs, such as the famous Birkhoff's theorem (HSP theorem) for algebraic theories. As a first step, we have shown the following result: for an equational system $\mathbb{S} = \mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R$,

- $\mathbb{S}$-algebras are closed under homomorphic images, if the endofunctor Γ preserves epimorphisms.
- $\mathbb{S}$-algebras are closed under subalgebras.
- $\mathbb{S}$-algebras are closed under products.

However, at present we do not know whether these properties generally characterize the classes of algebras for ESs (with Γ preserving epimorphisms).

Although we have no general completeness result for TEL, TEL turned out to be complete for all our concrete examples of TES. Thus, we are interested in classes of TESs for which TEL is complete, or whether there are additional sound rules that make TEL generally complete. In particular, we would like to investigate whether TEL gives rise to a complete logic for enriched algebraic theories.

As an important application of TES, we are interested in developing equational logic and rewriting system for second-order abstract syntax [Fiore 2008]. Indeed, second-order abstract syntax induces an associated TES, as it is represented by the monad induced from free Σ-monoids (see Section 5.2) on the presheaf category $\mathbf{Set}^{\mathbb{F}}$ for appropriate endofunctors Σ, where $\mathbb{F}$ denotes the (essentially small) category of finite sets and functions. Thus, from the TEL for the associated TES, we can extract a sound syntactic equational logic for the second-order abstract syntax, which we expect to be logically complete. We will also try to synthesize a sound and complete rewriting system following our methodology proposed in Section 7.2, which we expect to be a rewriting system similar to the Combinatory Reduction System (CRS) of Klop [1980]. This will be further investigated with Fiore and published elsewhere.

In the context of the enriched algebraic theories of Kelly and Power [1993], one may also consider the categorical presentation of term rewriting via coinserters of Ghani and Lüth [2003] in the setting of algebraic theories on the category of preorders. In this vein, we have developed the concepts of Equational Rewrite System (ERS) and Term Equational Rewrite System (TERS), generalizing the concepts of ES and TES into an abstract-rewriting enriched setting. More precisely, we enrich all notions for ES and TES with the category $\mathscr{R}$ of abstract rewrite systems whose objects are binary relations on sets, called *abstract rewrite systems* or simply *rewrites*, and whose morphisms are functions between underlying sets that preserve the associated binary relations. The concept of rewrite (*i.e.*, arbitrary binary relations on sets), rather than that of preorder, captures the notion of single-step rewrite relation, which is not reflexive nor transitive.

An important advantage of rewrites over preorders is that not only confluence but also normalization (or termination) can be considered. We have also developed a theory of free construction for ERSs, and a sound rewrite logic, called Term Equational Rewrite Logic (TERL), and an internal completeness result for TERSs. This is joint work with Fiore and details will appear elsewhere.

In their setting of abstract rewriting, Abbott et al. [2005] presented abstract conditions for modularity of confluence and of strong normalization. Similarly, we are interested in seeking abstract conditions for such properties in our more general setting of TERS. As a first step, generalizing the idea of Abbott et al. [2005] towards modular properties of constructor-sharing term rewriting systems of Ohlebusch [1994], we have found some abstract conditions for modularity of confluence and of strong normalization of TERSs with shared constructors. This joint work with Fiore will also be published elsewhere.

Appendix A

Enriched categories induced from actions of monoidal categories

For a monoidal category $\mathscr{V} = (\mathscr{V}, \cdot, I, \alpha, \lambda, \rho)$, every right-closed $\mathscr{V}$-action $(\mathscr{C}, *, \widetilde{\alpha}, \widetilde{\lambda})$ with right-homs $\underline{\mathscr{C}}(C, -) : \mathscr{C} \to \mathscr{V}$ and evaluation maps $\epsilon_X^C : \underline{\mathscr{C}}(C, X) * C \to X$ induces the $\mathscr{V}$-enriched category consisting of

- hom-objects $\underline{\mathscr{C}}(A, B) \in \mathscr{V}$ for all objects $A, B \in \mathscr{C}$;
- identity maps $j_A : I \to \underline{\mathscr{C}}(A, A)$ for all objects $A \in \mathscr{C}$ given by the transpose of the map $\widetilde{\lambda}_A : I * A \to A$; and
- composition maps $M_{A,B,C} : \underline{\mathscr{C}}(B, C) \cdot \underline{\mathscr{C}}(A, B) \to \underline{\mathscr{C}}(A, C)$ for all object $A, B, C \in \mathscr{C}$ given by the transpose of the composite

$$\big(\underline{\mathscr{C}}(B, C) \cdot \underline{\mathscr{C}}(A, B)\big) * A \xrightarrow{\widetilde{\alpha}_{\underline{\mathscr{C}}(B,C), \underline{\mathscr{C}}(A,B), A}} \underline{\mathscr{C}}(B, C) * \big(\underline{\mathscr{C}}(A, B) * A\big)$$
$$\xrightarrow{\underline{\mathscr{C}}(B,C) * \epsilon_B^A} \underline{\mathscr{C}}(B, C) * B \xrightarrow{\epsilon_C^B} C$$

satisfying the unit and associativity axioms as shown below.

- The commutativity of the left unit axiom

$$\begin{tikzcd}
\underline{\mathscr{C}}(B, B) \cdot \underline{\mathscr{C}}(A, B) \arrow[rr, "M_{A,B,B}"] & & \underline{\mathscr{C}}(A, B) \\
I \cdot \underline{\mathscr{C}}(A, B) \arrow[u, "j_B \cdot \underline{\mathscr{C}}(A,B)"] \arrow[urr, "\lambda_{\underline{\mathscr{C}}(A,B)}"'] & &
\end{tikzcd}$$

  follows from that of its transpose:

$$\begin{tikzcd}
(\underline{\mathscr{C}}(B, B) \cdot \underline{\mathscr{C}}(A, B)) * A \arrow[r, "\widetilde{\alpha}_{\underline{\mathscr{C}}(B,B), \underline{\mathscr{C}}(A,B), A}"] & \underline{\mathscr{C}}(B, B) * (\underline{\mathscr{C}}(A, B) * A) \arrow[r, "\underline{\mathscr{C}}(B,B) * \epsilon_B^A"] & \underline{\mathscr{C}}(B, B) * B \arrow[r, "\epsilon_B^B"] & B \\
 & I * (\underline{\mathscr{C}}(A, B) * A) \arrow[u, "j_B * (\underline{\mathscr{C}}(A,B) * A)"] \arrow[r, "I * \epsilon_B^A"] \arrow[dr, "\widetilde{\lambda}_{\underline{\mathscr{C}}(A,B) * A}"] & I * B \arrow[u, "j_B * B"] \arrow[ur, "\widetilde{\lambda}_B"] & \\
(I \cdot \underline{\mathscr{C}}(A, B)) * A \arrow[uu, "(j_B \cdot \underline{\mathscr{C}}(A,B)) * A"] \arrow[ur, "\widetilde{\alpha}_{I, \underline{\mathscr{C}}(A,B), A}"] \arrow[rr, "\lambda_{\underline{\mathscr{C}}(A,B)} * A"] & & \underline{\mathscr{C}}(A, B) * A \arrow[uur, "\epsilon_B^A"'] &
\end{tikzcd}$$

- The commutativity of the right unit axiom

$$\begin{tikzcd}
\underline{\mathscr{C}}(A,B)\cdot\underline{\mathscr{C}}(A,A) \arrow[r, "M_{A,A,B}"] & \underline{\mathscr{C}}(A,B) \\
\underline{\mathscr{C}}(A,B)\cdot I \arrow[u, "\underline{\mathscr{C}}(A,B)\cdot j_A"] \arrow[ur, "\rho_{\underline{\mathscr{C}}(A,B)}"'] &
\end{tikzcd}$$

follows from that of its transpose:

$$\begin{tikzcd}
(\underline{\mathscr{C}}(A,B)\cdot\underline{\mathscr{C}}(A,A))\ast A \arrow[r, "\widetilde{\alpha}_{\underline{\mathscr{C}}(A,B),\underline{\mathscr{C}}(A,A),A}"] & \underline{\mathscr{C}}(A,B)\ast(\underline{\mathscr{C}}(A,A)\ast A) \arrow[r, "\underline{\mathscr{C}}(A,B)\ast\epsilon^A_A"] & \underline{\mathscr{C}}(A,B)\ast A \arrow[r, "\epsilon^A_B"] & B \\
& \underline{\mathscr{C}}(A,B)\ast(I\ast A) \arrow[u, "\underline{\mathscr{C}}(A,B)\ast(j_A\ast A)"] \arrow[ur, "\underline{\mathscr{C}}(A,B)\ast\widetilde{\lambda}_A"'] \arrow[dr, "\underline{\mathscr{C}}(A,B)\ast\widetilde{\lambda}_A"] & & \\
(\underline{\mathscr{C}}(A,B)\cdot I)\ast A \arrow[uu, "(\underline{\mathscr{C}}(A,B)\cdot j_A)\ast A"'] \arrow[ur, "\widetilde{\alpha}_{\underline{\mathscr{C}}(A,B),I,A}"] \arrow[rr, "\rho_{\underline{\mathscr{C}}(A,B)}\ast A"] & & \underline{\mathscr{C}}(A,B)\ast A \arrow[uu, equal] \arrow[uur, "\epsilon^A_B"'] &
\end{tikzcd}$$

- The commutativity of the associativity axiom

$$\begin{tikzcd}
(\underline{\mathscr{C}}(C,D)\cdot\underline{\mathscr{C}}(B,C))\cdot\underline{\mathscr{C}}(A,B) \arrow[rr, "\alpha_{\underline{\mathscr{C}}(C,D),\underline{\mathscr{C}}(B,C),\underline{\mathscr{C}}(A,B)}"] \arrow[d, "M_{B,C,D}\cdot\underline{\mathscr{C}}(A,B)"'] & & \underline{\mathscr{C}}(C,D)\cdot(\underline{\mathscr{C}}(B,C)\cdot\underline{\mathscr{C}}(A,B)) \arrow[d, "\underline{\mathscr{C}}(C,D)\cdot M_{A,B,C}"] \\
\underline{\mathscr{C}}(B,D)\cdot\underline{\mathscr{C}}(A,B) \arrow[dr, "M_{A,B,D}"'] & & \underline{\mathscr{C}}(C,D)\cdot\underline{\mathscr{C}}(A,C) \arrow[dl, "M_{A,C,D}"] \\
& \underline{\mathscr{C}}(A,D) &
\end{tikzcd}$$

follows from that of its transpose:

$$\begin{tikzcd}
((\underline{\mathscr{C}}(C,D)\cdot\underline{\mathscr{C}}(B,C))\cdot\underline{\mathscr{C}}(A,B))\ast A \arrow[rrrr, "\alpha_{\underline{\mathscr{C}}(C,D),\underline{\mathscr{C}}(B,C),\underline{\mathscr{C}}(A,B)}\ast A"] \arrow[d, "(M_{B,C,D}\cdot\underline{\mathscr{C}}(A,B))\ast A"'] \arrow[ddr, "\widetilde{\alpha}_{\underline{\mathscr{C}}(C,D)\cdot\underline{\mathscr{C}}(B,C),\underline{\mathscr{C}}(A,B),A}"] & & & & (\underline{\mathscr{C}}(C,D)\cdot(\underline{\mathscr{C}}(B,C)\cdot\underline{\mathscr{C}}(A,B)))\ast A \arrow[d, "(\underline{\mathscr{C}}(C,D)\cdot M_{A,B,C})\ast A"] \arrow[ddl, "\widetilde{\alpha}_{\underline{\mathscr{C}}(C,D),\underline{\mathscr{C}}(B,C)\cdot\underline{\mathscr{C}}(A,B),A}"'] \\
(\underline{\mathscr{C}}(B,D)\cdot\underline{\mathscr{C}}(A,B))\ast A \arrow[dd, "\widetilde{\alpha}_{\underline{\mathscr{C}}(B,D),\underline{\mathscr{C}}(A,B),A}"'] & & & & (\underline{\mathscr{C}}(C,D)\cdot\underline{\mathscr{C}}(A,C))\ast A \arrow[dd, "\widetilde{\alpha}_{\underline{\mathscr{C}}(C,D),\underline{\mathscr{C}}(A,C),A}"] \\
& (\underline{\mathscr{C}}(C,D)\cdot\underline{\mathscr{C}}(B,C))\ast(\underline{\mathscr{C}}(A,B)\ast A) \arrow[dl, "M_{B,C,D}\ast(\underline{\mathscr{C}}(A,B)\ast A)"'] \arrow[dd, "(\underline{\mathscr{C}}(C,D)\cdot\underline{\mathscr{C}}(B,C))\ast\epsilon^A_B"] \arrow[dr, "\widetilde{\alpha}_{\underline{\mathscr{C}}(C,D),\underline{\mathscr{C}}(B,C),\underline{\mathscr{C}}(A,B)\ast A}"] & & \underline{\mathscr{C}}(C,D)\ast((\underline{\mathscr{C}}(B,C)\cdot\underline{\mathscr{C}}(A,B))\ast A) \arrow[dl, "\underline{\mathscr{C}}(C,D)\ast\widetilde{\alpha}_{\underline{\mathscr{C}}(B,C),\underline{\mathscr{C}}(A,B),A}"'] \arrow[dr, "\underline{\mathscr{C}}(C,D)\ast(M_{A,B,C}\ast A)"] & \\
\underline{\mathscr{C}}(B,D)\ast(\underline{\mathscr{C}}(A,B)\ast A) \arrow[dd, "\underline{\mathscr{C}}(B,D)\ast\epsilon^A_B"'] & & \underline{\mathscr{C}}(C,D)\ast(\underline{\mathscr{C}}(B,C)\ast(\underline{\mathscr{C}}(A,B)\ast A)) \arrow[dr, "\underline{\mathscr{C}}(C,D)\ast(\underline{\mathscr{C}}(B,C)\ast\epsilon^A_B)"] & & \underline{\mathscr{C}}(C,D)\ast(\underline{\mathscr{C}}(A,C)\ast A) \arrow[dd, "\underline{\mathscr{C}}(C,D)\ast\epsilon^A_C"] \\
& (\underline{\mathscr{C}}(C,D)\cdot\underline{\mathscr{C}}(B,C))\ast B \arrow[rr, "\widetilde{\alpha}_{\underline{\mathscr{C}}(C,D),\underline{\mathscr{C}}(B,C),B}"'] \arrow[dl, "M_{B,C,D}\ast B"] & & \underline{\mathscr{C}}(C,D)\ast(\underline{\mathscr{C}}(B,C)\ast B) \arrow[dr, "\underline{\mathscr{C}}(C,D)\ast\epsilon^B_C"'] & \\
\underline{\mathscr{C}}(B,D)\ast B \arrow[drr, "\epsilon^B_D"] & & & & \underline{\mathscr{C}}(C,D)\ast C \arrow[dll, "\epsilon^C_D"'] \\
& & D & &
\end{tikzcd}$$

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