Categorical Equational Systems: Algebraic Models and Equational Reasoning

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Summary

We introduce two abstract notions of equational algebraic system, called *Equational System (ES)* and *Term Equational System (TES)*, in order to achieve sufficient expressivity as needed in modern applications in computer science. These generalize the classical concept of (enriched) algebraic theory of Kelly and Power [1993]. We also develop a theory for constructing free algebras for ESs and a theory of equational reasoning for TESs.

In Part I, we introduce the general abstract, yet practical, concept of equational system and develop finitary and transfinitary conditions under which we give an explicit construction of free algebras for ESs. This free construction extends the well-known construction of free algebras for ω-cocontinuous endofunctors to an equational setting, capturing the intuition that free algebras consist of freely constructed terms quotiented by given equations and congruence rules. We further show the monadicity and cocompleteness of categories of algebras for ESs under the finitary and transfinitary conditions. To illustrate the expressivity of equational systems, we exhibit various examples including two modern applications, the Σ-monoids of Fiore et al. [1999] and the π-algebras of Stark [2005].

In Part II, we introduce the more concrete notion of term equational system, which is obtained by specializing the concept of equational system, but remains more general than that of enriched algebraic theory. We first develop a sound logical deduction system, called *Term Equational Logic (TEL)*, for equational reasoning about algebras of TESs. Then, to pursue a complete logic, we give an *internal completeness* result, from which together with the explicit construction of free algebras one can typically synthesize sound and complete rewriting-style equational logics. To exemplify this scenario, we give two applications: multi-sorted algebraic theories and nominal equational theories of Clouston and Pitts [2007] and of Gabbay and Mathijssen [2007].

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